Further Math P2
Topical
2014-2022

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HYPERBOLIC FUNCTIONS

Q1.

2 **(a)** Starting from the definitions of cosh and sinh in terms of exponentials, prove that

$$\cosh 2x = 2\sinh^2 x + 1.$$ [3]

(b) Find the set of values of k for which $\cosh 2x = k\sinh x$ has two distinct real roots. [5]

HYPERBOLIC FUNCTIONS
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Q1.

Question	Answer	Marks	Guidance
2(a)	$\cosh x = \frac{1}{2}\left(e^{x} + e^{-x}\right) \quad \sinh x = \frac{1}{2}\left(e^{x} - e^{-x}\right)$	**B1**	
	$\frac{1}{2}\left(e^{x} - e^{-x}\right)^2 + 1 = \frac{1}{2}\left(e^{2x} - e^{-2x}\right) = \cosh 2x$	**M1 A1**	Expands, AG.
		3	
2(b)	$2\sinh^2 x - k\sinh x + 1 = 0$	**M1 A1**	Applies identity.
	$k^2 - 8 > 0$	**M1 A1**	Sets discriminant positive.
	$k < -\sqrt{8},\ k > \sqrt{8}$	**A1**	
		5	

Matrices 2

Q1.

8 **(a)** Find the value of a for which the system of equations

$$\begin{aligned} 3x + ay &= 0, \\ 5x - y &= 0, \\ x + 3y + 2z &= 0, \end{aligned}$$

does not have a unique solution. [2]

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The matrix **A** is given by

$$\mathbf{A} = \begin{pmatrix} 3 & 0 & 0 \\ 5 & -1 & 0 \\ 1 & 3 & 2 \end{pmatrix}.$$

(b) Find a matrix **P** and a diagonal matrix **D** such that $\mathbf{A}^2 = \mathbf{PDP}^{-1}$. [7]

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(c) Use the characteristic equation of $\mathbf{A}$ to show that

$$(\mathbf{A}+6\mathbf{I})^2 = \mathbf{A}^4(\mathbf{A}+b\mathbf{I})^2,$$

where b is an integer to be determined. [4]

Q2.

3 The matrix **A** is given by

$$\mathbf{A} = \begin{pmatrix} 6 & -9 & 5 \\ 5 & -8 & 5 \\ 1 & -1 & 2 \end{pmatrix}.$$

(a) Find the eigenvalues of **A**. [4]

(b) Use the characteristic equation of $\mathbf{A}$ to show that $\mathbf{A}^{-1} = p\mathbf{A}^2 + q\mathbf{I}$, where p and q are constants to be determined. [3]

1 **(a)** Given that a is an integer, show that the system of equations

$$\begin{aligned} ax + 3y + z &= 14, \\ 2x + y + 3z &= 0, \\ -x + 2y - 5z &= 17, \end{aligned}$$

has a unique solution and interpret this situation geometrically. [4]

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(b) Find the value of a for which $x = 1,\ y = 4,\ z = -2$ is the solution to the system of equations in part **(a)**. [1]

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Q4.

6 The matrix **A** is given by

$$\mathbf{A} = \begin{pmatrix} 5 & -\frac{22}{3} & 8 \\ 0 & -6 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(a) Find a matrix **P** and a diagonal matrix **D** such that $\mathbf{A}^2 = \mathbf{PDP}^{-1}$. [7]

(b) Use the characteristic equation of $\mathbf{A}$ to find $\mathbf{A}^3$. [4]

Q5.

8 **(a)** Find the value of a for which the system of equations

$$\begin{aligned} 13x + 18y - 28z &= 0, \\ -4x - ay + 8z &= 0, \\ 2x + 6y - 5z &= 0, \end{aligned}$$

does not have a unique solution. [2]

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The matrix **A** is given by

$$\mathbf{A} = \begin{pmatrix} 13 & 18 & -28 \\ -4 & -1 & 8 \\ 2 & 6 & -5 \end{pmatrix}.$$

(b) Find the eigenvalue of **A** corresponding to the eigenvector $\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$. [1]

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(c) Find a matrix **P** and a diagonal matrix **D** such that $\mathbf{A} = \mathbf{PDP}^{-1}$. [8]

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(d) Use the characteristic equation of $\mathbf{A}$ to find $\mathbf{A}^{-1}$ in terms of $\mathbf{A}$. [2]

Q6.

2 The matrix **A** is given by

$$\mathbf{A} = \begin{pmatrix} -1 & 2 & 12 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

Use the characteristic equation of **A** to show that

$$\mathbf{A}^4 = p\mathbf{A}^2 + q\mathbf{I},$$

where p and q are integers to be determined. [6]

Q7.

6 The matrix **P** is given by

$$\mathbf{P} = \begin{pmatrix} 1 & 6 & 6 \\ 0 & 2 & 6 \\ 0 & 0 & -3 \end{pmatrix}.$$

(a) Use the characteristic equation of **P** to find $\mathbf{P}^{-1}$. [5]

(b) Find the matrix **A** such that

$$\mathbf{P}^{-1}\mathbf{AP} = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 6 \end{pmatrix}. \qquad [4]$$

(c) State the eigenvalues and corresponding eigenvectors of $\mathbf{A}^3$. [2]

Q8.

8 **(a)** Find the values of a for which the system of equations

$$\begin{aligned} 3x + y + z &= 0, \\ ax + 6y - z &= 0, \\ ay - 2z &= 0, \end{aligned}$$

does not have a unique solution. [3]

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The matrix **A** is given by

$$\mathbf{A} = \begin{pmatrix} 3 & 1 & 1 \\ 0 & 6 & -1 \\ 0 & 0 & -2 \end{pmatrix}.$$

(b) Use the characteristic equation of **A** to find the inverse of $\mathbf{A}^2$. [4]

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(c) Find a matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{A}^5 = \mathbf{PDP}^{-1}$. [7]

Q9.

3 The matrix **A** is given by

$$\mathbf{A} = \begin{pmatrix} 5 & -1 & 7 \\ 0 & 6 & 0 \\ 7 & 7 & 5 \end{pmatrix}.$$

(a) Find the eigenvalues of **A**. [4]

(b) Use the characteristic equation of **A** to find $\mathbf{A}^{-1}$. [4]

Q10.

9 It is given that **e** is an eigenvector of the matrix **A**, with corresponding eigenvalue λ.

(i) Show that **e** is an eigenvector of $\mathbf{A}^2$, with corresponding eigenvalue λ^2. [2]

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The matrices **A** and **B** are given by

$$\mathbf{A} = \begin{pmatrix} n & 1 & 3 \\ 0 & 2n & 0 \\ 0 & 0 & 3n \end{pmatrix} \quad \text{and} \quad \mathbf{B} = (\mathbf{A} + n\mathbf{I})^2,$$

where **I** is the 3×3 identity matrix and n is a non-zero integer.

(ii) Find, in terms of n, a non-singular matrix **P** and a diagonal matrix **D** such that $\mathbf{B} = \mathbf{PDP}^{-1}$. [8]

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Q11.

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A 3×3 matrix **A** has distinct eigenvalues 2, 1, 3, with corresponding eigenvectors

$$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} -1 \\ 0 \\ b \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

respectively, where b is a positive constant.

(i) Find **A** in terms of b. [9]

(ii) Find $\mathbf{A}^{-1}\begin{pmatrix} 0 \\ 2 \\ -2 \end{pmatrix}$. [2]

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(iii) It is given that

$$\mathbf{A}^n\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{A}^n\begin{pmatrix} -1 \\ 0 \\ b \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ b^{-1} \end{pmatrix}.$$

Find the values of n and b. [3]

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Q12.

8 The matrix **M** is defined by

$$\mathbf{M} = \begin{pmatrix} 2 & m & 1 \\ 0 & m & 7 \\ 0 & 0 & 1 \end{pmatrix},$$

where $m \neq 0, 1, 2$.

(i) Find a matrix **P** and a diagonal matrix **D** such that $\mathbf{M} = \mathbf{PDP}^{-1}$. [7]

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(ii) Find $\mathbf{M}^7\mathbf{P}$. [3]

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Q13.

10 The matrix **A** is defined by

$$\mathbf{A} = \begin{pmatrix} 1 & 5 & 1 \\ 1 & -2 & -2 \\ 2 & 3 & \theta \end{pmatrix}.$$

(i) (a) Find the rank of **A** when $\theta \neq -1$. [3]

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(b) Find the rank of **A** when $\theta = -1$. [1]

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Consider the system of equations

$$\begin{aligned} x + 5y + z &= -1, \\ x - 2y - 2z &= 0, \\ 2x + 3y + \theta z &= \theta. \end{aligned}$$

(ii) Solve the system of equations when $\theta \neq -1$. [3]

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(iii) Find the general solution when $\theta = -1$. [3]

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(iv) Show that if $\theta = -1$ and $\phi \neq -1$ then $\mathbf{Ax} = \begin{pmatrix} -1 \\ 0 \\ \phi \end{pmatrix}$ has no solution. [2]

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Q14.

It is given that **e** is an eigenvector of the matrix **A**, with corresponding eigenvalue λ.

(i) Write down another eigenvector of **A** corresponding to λ. [1]

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(ii) Write down an eigenvector and corresponding eigenvalue of $\mathbf{A}^n$, where n is a positive integer. [2]

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Let $\mathbf{A} = \begin{pmatrix} 3 & 0 & 0 \\ 2 & 7 & 0 \\ 4 & 8 & 1 \end{pmatrix}$.

(iii) Find a matrix **P** and a diagonal matrix **D** such that $\mathbf{A}^n = \mathbf{PDP}^{-1}$. [7]

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(iv) Determine the set of values of the real constant k such that

$$\sum_{n=1}^{\infty} k^n(\mathbf{A}^n - k\mathbf{A}^{n+1}) = k\mathbf{A}. \qquad [4]$$

Q15.

5 It is given that **e** is an eigenvector of the matrix **A** with corresponding eigenvalue λ.

(i) Show that **e** is an eigenvector of $\mathbf{A}^3$ and state the corresponding eigenvalue. [3]

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It is given that

$$\mathbf{A} = \begin{pmatrix} 2 & 0 \\ -1 & 3 \end{pmatrix}.$$

(ii) Find a matrix **P** and a diagonal matrix **D** such that

$$\mathbf{A}^3 + \mathbf{I} = \mathbf{PDP}^{-1},$$

where **I** is the 2×2 identity matrix. [5]

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Q16.

5 It is given that λ is an eigenvalue of the matrix **A** with **e** as a corresponding eigenvector, and μ is an eigenvalue of the matrix **B** for which **e** is also a corresponding eigenvector.

(i) Show that $\lambda + \mu$ is an eigenvalue of the matrix $\mathbf{A} + \mathbf{B}$ with **e** as a corresponding eigenvector. [2]

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The matrix **A**, given by

$$\mathbf{A} = \begin{pmatrix} 2 & 0 & 1 \\ -1 & 2 & 3 \\ 1 & 0 & 2 \end{pmatrix}$$

has $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 4 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ as eigenvectors.

(ii) Find the corresponding eigenvalues. [3]

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The matrix **B** has eigenvalues 4, 5 and 1 with corresponding eigenvectors $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 4 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ respectively.

(iii) Find a matrix **P** and a diagonal matrix **D** such that $(\mathbf{A} + \mathbf{B})^3 = \mathbf{PDP}^{-1}$. [3]

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Q17.

2 It is given that

$$\mathbf{A} = \begin{pmatrix} 2 & 3 & 1 \\ 0 & -2 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

(i) Find the eigenvalue of $\mathbf{A}$ corresponding to the eigenvector $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. [1]

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(ii) Write down the negative eigenvalue of $\mathbf{A}$ and find a corresponding eigenvector. [3]

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(iii) Find an eigenvalue and a corresponding eigenvector of the matrix $\mathbf{A} + \mathbf{A}^6$. [2]

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Q18.

4 **(i)** Find the value of k for which the set of linear equations

$$\begin{aligned} x + 3y + kz &= 4, \\ 4x - 2y - 10z &= -5, \\ x + y + 2z &= 1, \end{aligned}$$

has no unique solution. [3]

(ii) For this value of k, find the set of possible solutions, giving your answer in the form

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{a} + t\mathbf{b},$$

where $\mathbf{a}$ and $\mathbf{b}$ are vectors and t is a scalar. [3]

Q19.

5 The matrix **A**, given by

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & -2 \\ 6 & 4 & -6 \\ 6 & 5 & -7 \end{pmatrix},$$

has eigenvalues 1, −1 and −2.

(i) Find a set of corresponding eigenvectors. [4]

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(ii) The matrix **B** is given by $\mathbf{B} = \mathbf{A} - 2\mathbf{I}$, where **I** is the 3×3 identity matrix. Write down the eigenvalues of **B**, and state a set of corresponding eigenvectors. [2]

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Q20.

10 The matrix **A** is given by

$$\mathbf{A} = \begin{pmatrix} 6 & -8 & 7 \\ 7 & -9 & 7 \\ 6 & -6 & 5 \end{pmatrix}.$$

(i) Given that $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ is an eigenvector of **A**, find the corresponding eigenvalue. [2]

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(ii) Given also that -1 is an eigenvalue of **A**, find a corresponding eigenvector. [2]

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(iii) It is given that the determinant of **A** is equal to the product of the eigenvalues of **A**. Use this result to find the third eigenvalue of **A**, and find also a corresponding eigenvector. [3]

(iv) Write down matrices $\mathbf{P}$ and $\mathbf{D}$ such that $\mathbf{P}^{-1}\mathbf{AP} = \mathbf{D}$, where $\mathbf{D}$ is a diagonal matrix, and hence find the matrix $\mathbf{A}^n$ in terms of n, where n is a positive integer. [6]

Q21.

11 **(i)** The vector **e** is an eigenvector of the matrix **A**, with corresponding eigenvalue λ, and is also an eigenvector of the matrix **B**, with corresponding eigenvalue μ. Show that **e** is an eigenvector of the matrix **AB** with corresponding eigenvalue $\lambda\mu$. [3]

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(ii) Find the eigenvalues and corresponding eigenvectors of the matrix **A**, where

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 3 \\ 3 & 2 & -3 \\ 1 & 1 & 2 \end{pmatrix}. \qquad [6]$$

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(iii) The matrix $\mathbf{B}$, where

$$\mathbf{B} = \begin{pmatrix} 3 & 6 & 1 \\ 1 & -2 & -1 \\ 6 & 6 & -2 \end{pmatrix},$$

has eigenvectors $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$. Find the eigenvalues of the matrix $\mathbf{AB}$, and state corresponding eigenvectors. [4]

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Q22.

10 Write down the eigenvalues of the matrix **A**, where

$$\mathbf{A} = \begin{pmatrix} -2 & 1 & -1 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{pmatrix},$$

and find corresponding eigenvectors. [4]

Find a matrix **P** and a diagonal matrix **D** such that $\mathbf{P}^{-1}\mathbf{AP} = \mathbf{D}$, and hence find the matrix $\mathbf{A}^n$, where n is a positive integer. [8]

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Q23.

3 Find the two values of the constant k for which the equations

$$\begin{aligned} kx + y + z &= 2, \\ x + ky + z &= -1, \\ x + y + kz &= -1, \end{aligned}$$

have no unique solution. [4]

Show that, for one of these values of k, the equations have no solution, and solve the equations for the other value of k. [3]

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Q24.

11 It is given that 1 and 4 are eigenvalues of the matrix **A**, where

$$\mathbf{A} = \begin{pmatrix} 1 & -3 & -3 \\ -8 & 6 & -3 \\ 8 & -2 & 7 \end{pmatrix}.$$

Find eigenvectors corresponding to each of these eigenvalues. [3]

Given further that $\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$ is an eigenvector of **A**, find the corresponding eigenvalue. [2]

Write down matrices **P** and **D** such that $\mathbf{P}^{-1}\mathbf{AP} = \mathbf{D}$, where **D** is a diagonal matrix, and find $\mathbf{P}^{-1}$. [5]

Write down a matrix **C** such that $\mathbf{C}^2 = \mathbf{D}$, and deduce a matrix **B** such that $\mathbf{B}^2 = \mathbf{A}$. [4]

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Q25.

3 Find a matrix **A** whose eigenvalues are −1, 1, 2 and for which corresponding eigenvectors are

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix},$$

respectively. [7]

Q26.

2 Find the value of the constant k for which the system of equations

$$\begin{aligned} 2x - 3y + 4z &= 1, \\ 3x - y &= 2, \\ x + 2y + kz &= 1, \end{aligned}$$

does not have a unique solution. [2]

For this value of k, solve the system of equations. [4]

Q27.

10 The matrix **A** is given by

$$\mathbf{A} = \begin{pmatrix} 2 & 2 & -3 \\ 2 & 2 & 3 \\ -3 & 3 & 3 \end{pmatrix}.$$

The matrix **A** has an eigenvector $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$. Find the corresponding eigenvalue. [2]

The matrix **A** also has eigenvalues 4 and 6. Find corresponding eigenvectors. [3]

Hence find a matrix such that **A** = **D** $^{-1}$, where **D** is a diagonal matrix which is to be determined. [2]

The matrix **B** is such that $\mathbf{B} = \mathbf{QAQ}^{-1}$, where

$$\mathbf{Q} = \begin{pmatrix} 4 & 11 & 5 \\ 1 & 4 & 2 \\ 1 & 2 & 1 \end{pmatrix}.$$

By using the expression **D** $^{-1}$ for **A**, find the set of eigenvalues and a corresponding set of eigenvectors for **B**. [5]

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Q28.

11 One of the eigenvalues of the matrix **M**, where

$$\mathbf{M} = \begin{pmatrix} 3 & -4 & 2 \\ -4 & \alpha & 6 \\ 2 & 6 & -2 \end{pmatrix},$$

is −9. Find the value of α. [3]

Find

(i) the other two eigenvalues, λ_1 and λ_2, of **M**, where $\lambda_1 > \lambda_2$, [5]

(ii) corresponding eigenvectors for all three eigenvalues of **M**. [3]

It is given that $= a\mathbf{e}_1 + b\mathbf{e}_2$, where $\mathbf{e}_1$ and $\mathbf{e}_2$ are eigenvectors of **M** corresponding to the eigenvalues λ_1 and λ_2 respectively, and a and b are scalar constants. Show that $\mathbf{M} = p\mathbf{e}_1 + q\mathbf{e}_2$, expressing p and q in terms of a and b. [3]

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Q29.

6 The matrix **A**, where

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 10 & -7 & 10 \\ 7 & -5 & 8 \end{pmatrix},$$

has eigenvalues 1 and 3. Find corresponding eigenvectors. [3]

It is given that $\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$ is an eigenvector of **A**. Find the corresponding eigenvalue. [2]

Find a diagonal matrix **D** and matrices and $^{-1}$ such that $^{-1}\mathbf{A}$ = **D**. [5]

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Q30.

9 The matrix **M**, where

$$\mathbf{M} = \begin{pmatrix} -2 & 2 & 2 \\ 2 & 1 & 2 \\ -3 & -6 & -7 \end{pmatrix},$$

has an eigenvector $\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$. Find the corresponding eigenvalue. [2]

It is given that if the eigenvalues of a general 3×3 matrix **A**, where

$$\mathbf{A} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix},$$

are λ_1, λ_2 and λ_3 then

$$\lambda_1 + \lambda_2 + \lambda_3 = a + e + i$$

and

the determinant of **A** has the value $\lambda_1\lambda_2\lambda_3$.

Use these results to find the other two eigenvalues of the matrix **M**, and find corresponding eigenvectors. [8]

Q31.

8 It is given that λ is an eigenvalue of the non-singular square matrix **A**, with corresponding eigenvector **e**. Show that λ^{-1} is an eigenvalue of $\mathbf{A}^{-1}$ for which **e** is a corresponding eigenvector. [2]

Deduce that $\lambda + \lambda^{-1}$ is an eigenvalue of $\mathbf{A} + \mathbf{A}^{-1}$. [1]

It is given that 1 is an eigenvalue of the matrix **A**, where

$$\mathbf{A} = \begin{pmatrix} 2 & 0 & 1 \\ -1 & 2 & 3 \\ 1 & 0 & 2 \end{pmatrix}.$$

Find a corresponding eigenvector. [2]

It is also given that $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ are eigenvectors of the matrix **A**. Find the corresponding eigenvalues. [2]

Hence find a matrix and a diagonal matrix **D** such that

$$\left(\mathbf{A} + \mathbf{A}^{-1}\right)^3 = \mathbf{D}^{-1}. \quad [4]$$

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Q32.

5 Find the value of a for which the system of equations

$$\begin{aligned} x - \ \ y + 2z &= 4, \\ x + ay - 3z &= b, \\ x - \ \ y + 7z &= 13, \end{aligned}$$

where a and b are constants, has no unique solution. [3]

Taking a as the value just found,

(i) find the general solution in the case $b = -5$, [4]

(ii) interpret the situation geometrically in the case $b \neq -5$. [1]

Q33.

11 The square matrix **A** has λ as an eigenvalue with **e** as a corresponding eigenvector. Show that if **A** is non-singular then

(i) $\lambda \neq 0$, [2]

(ii) the matrix $\mathbf{A}^{-1}$ has λ^{-1} as an eigenvalue with **e** as a corresponding eigenvector. [2]

The 3×3 matrices **A** and **B** are given by

$$\mathbf{A} = \begin{pmatrix} -2 & 2 & -4 \\ 0 & -1 & 5 \\ 0 & 0 & 3 \end{pmatrix} \quad \text{and} \quad \mathbf{B} = (\mathbf{A} + 3\mathbf{I})^{-1},$$

where **I** is the 3×3 identity matrix. Find a non-singular matrix **P**, and a diagonal matrix **D**, such that $\mathbf{B} = \mathbf{PDP}^{-1}$. [10]

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MATRICES 2
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Q1.

Question	Answer	Marks	Guidance
8(a)	$\begin{vmatrix} 3 & a & 0 \\ 5 & -1 & 0 \\ 1 & 3 & 2 \end{vmatrix} = 0 \Rightarrow -10a - 6 = 0$	M1	Sets determinant equal to zero.
	$a = -\frac{3}{5}$	A1	
		2	
8(b)	Eigenvalues of **A** are 3, -1 and 2.	B1	Lower diagonal matrix or characteristic equation.
	$\lambda = 3$: $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 5 & -4 & 0 \\ 1 & 3 & -1 \end{vmatrix} = \begin{pmatrix} 4 \\ 5 \\ 19 \end{pmatrix}$	M1 A1	Uses vector product (or equations) to find corresponding eigenvectors.
	$\lambda = -1$: $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 0 & 0 \\ 1 & 3 & 3 \end{vmatrix} = \begin{pmatrix} 0 \\ -12 \\ 12 \end{pmatrix} \sim \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$	A1	
	$\lambda = 2$: $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 1 & 3 & 0 \end{vmatrix} = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} \sim \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$	A1	
	Thus $\mathbf{P} = \begin{pmatrix} 4 & 0 & 0 \\ 5 & -1 & 0 \\ 19 & 1 & 1 \end{pmatrix}$ and $\mathbf{D} = \begin{pmatrix} 9 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}$	M1 A1	Or correctly matched permutations of columns.
		7	

Question	Answer	Marks	Guidance
8(c)	$(\lambda-3)(\lambda+1)(\lambda-2) = \lambda^3 - 4\lambda^2 + \lambda + 6 = 0$	B1	Characteristic equation.
	$\mathbf{A} + 6\mathbf{I} = -\mathbf{A}^3 + 4\mathbf{A}^2$	M1	Substitutes for **A** and makes $\mathbf{A} + 6\mathbf{I}$ the subject.
	$(\mathbf{A} + 6\mathbf{I})^2 = (-\mathbf{A}^3 + 4\mathbf{A}^2)^2 = \mathbf{A}^4(\mathbf{A} - 4\mathbf{I})^2$	M1 A1	Squares and factorises.
		4	

Q2.

Question	Answer	Marks	Guidance
3(a)	$\begin{vmatrix} 6-\lambda & -9 & 5 \\ 5 & -8-\lambda & 5 \\ 1 & -1 & 2-\lambda \end{vmatrix} = 0$	M1	Sets determinant equal to zero.
	$\lambda^3 - 7\lambda + 6 = 0$	M1	Expands.
	$\lambda = 1, 2, -3$	A1 A1	Award A1A0 for 2 correct solutions, A0A0 for 1 correct solution.
		4	

Question	Answer	Marks	Guidance
3(b)	$\mathbf{A}^3 - 7\mathbf{A} + 6\mathbf{I} = 0$	**B1**	States that **A** satisfies its characteristic equation.
	$\mathbf{A}^2 - 7\mathbf{I} + 6\mathbf{A}^{-1} = 0$	**M1**	Multiplies through by $\mathbf{A}^{-1}$.
	$\mathbf{A}^{-1} = -\frac{1}{6}\mathbf{A}^2 + \frac{7}{6}\mathbf{I}$	**A1**	
		3	

Q3.

Question	Answer	Marks	Guidance
1(a)	$\begin{vmatrix} a & 3 & 1 \\ 2 & 1 & 3 \\ -1 & 2 & -5 \end{vmatrix} = -11a + 26$	**M1 A1**	Finds determinant. If solving equations x, y, z in terms in a.
	$\det \mathbf{A} \neq 0$ leads to unique solution.	**A1**	States that $\det \mathbf{A} \neq 0$. (Since a is an integer.) (M1 A0 A0 if one of x, y, z found in terms of a.) (M1 A1 A0 if two of x, y, z found in terms of a.)
	The three planes intersect at a single point.	**B1**	
		4	
1(b)	$a = 4$	**B1**	
		1	

Q4.

Question	Answer	Marks	Guidance
6(a)	Eigenvalues of **A** are 5, -6 and 1.	**B1**	Upper diagonal matrix or characteristic equation.
	$\lambda=5:\ \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 0 & -\frac{22}{3} & 8\\ 0 & -11 & 0\end{vmatrix}=\begin{pmatrix}88\\0\\0\end{pmatrix}\sim\begin{pmatrix}1\\0\\0\end{pmatrix}$ $\lambda=-6:\ \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 11 & -\frac{22}{3} & 8\\ 0 & 0 & 7\end{vmatrix}=\begin{pmatrix}-\frac{154}{3}\\-77\\0\end{pmatrix}\sim\begin{pmatrix}2\\3\\0\end{pmatrix}$ $\lambda=1:\ \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 4 & -\frac{22}{3} & 8\\ 0 & -7 & 0\end{vmatrix}=\begin{pmatrix}56\\0\\-28\end{pmatrix}\sim\begin{pmatrix}2\\0\\-1\end{pmatrix}$	**M1** **A1 A1 A1**	Uses vector product (or equations) to find corresponding Eigenvectors. A1 for each correct Eigenvector.
	Thus $\mathbf{P}=\begin{pmatrix}1&2&2\\0&3&0\\0&0&-1\end{pmatrix}$ and $\mathbf{D}=\begin{pmatrix}25&0&0\\0&36&0\\0&0&1\end{pmatrix}$	**M1 A1**	Or correctly matched permutations of columns. (Accept scalar multiples of the eigenvectors shown here.) **P** must have at least two non-zero columns.
		7	
6(b)	$(\lambda-5)(\lambda+6)(\lambda-1)=\lambda^3-31\lambda+30=0$ $=0$	**B1**	Finds characteristic equation.
	$\mathbf{A}^3-31\mathbf{A}+30\mathbf{I}=\mathbf{0}\Rightarrow\mathbf{A}^3=31\mathbf{A}-30\mathbf{I}$	**M1**	Finds $\mathbf{A}^3$ in terms of **A**. Allow missing **I**.
	$\mathbf{A}^3=\begin{pmatrix}125&-\frac{682}{3}&248\\0&-216&0\\0&0&1\end{pmatrix}$	**M1A1**	Substitutes for **A**.
		4	

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Q5.

8(a)	$\begin{vmatrix} 13 & 18 & -28 \\ -4 & -a & 8 \\ 2 & 6 & -5 \end{vmatrix} = 9a - 24$	**M1**	Finds determinant.
	$a = \frac{8}{3}$	**A1**	Accept $\frac{24}{9}$.
		2	
8(b)	$\begin{pmatrix} 13 & 18 & -28 \\ -4 & -1 & 8 \\ 2 & 6 & -5 \end{pmatrix}\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \\ -1 \end{pmatrix} \Rightarrow \lambda = -1.$	**B1**	
		1	

Question	Answer	Marks	Guidance
8(c)	$\begin{vmatrix} 13-\lambda & 18 & -28 \\ -4 & -1-\lambda & 8 \\ 2 & 6 & -5-\lambda \end{vmatrix} = 0$	**B1**	Sets $\det(\mathbf{A}-\lambda\mathbf{I}) = 0$.
	$-\lambda^3 + 7\lambda^2 - 7\lambda - 15 = 0 \Rightarrow (\lambda+1)(\lambda-3)(\lambda-5) = 0$	**M1**	Expands determinant and factorises.
	$\lambda = -1, \quad \lambda = 3, \quad \lambda = 5$	**A1**	
	$\lambda = 3$: $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -4 & -4 & 8 \\ 2 & 6 & -8 \end{vmatrix} = \begin{pmatrix} -16 \\ -16 \\ -16 \end{pmatrix} \sim \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ $\lambda = 5$: $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -4 & -6 & 8 \\ 2 & 6 & -10 \end{vmatrix} = \begin{pmatrix} 12 \\ -24 \\ -12 \end{pmatrix} \sim \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$	**M1** **A1 A1**	Uses vector product (or equations) to find corresponding eigenvectors. A1 for each correct Eigenvector.
	Thus $\mathbf{P} = \begin{pmatrix} 2 & 1 & -1 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix}$ and $\mathbf{D} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}$	**M1 A1**	Or correctly matched permutations of columns. Accept scalar multiples of eigenvectors. At least two (non-zero) Eigenvectors for M1.
		8	

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Question	Answer	Marks	Guidance
8(d)	$-\mathbf{A}^3+7\mathbf{A}^2-7\mathbf{A}-15\mathbf{I}=\mathbf{0}$	**M1**	Substitutes **A** into characteristic equation and multiplies through by $\mathbf{A}^{-1}$. Allow missing **I**.
	$\mathbf{A}^{-1}=\frac{1}{15}\left(-\mathbf{A}^2+7\mathbf{A}-7\mathbf{I}\right)$	**A1**	Accept with $\mathbf{A}=\begin{pmatrix}13 & 18 & -28\\ -4 & -1 & 8\\ 2 & 6 & -5\end{pmatrix}$ and $\mathbf{A}^2=\begin{pmatrix}41 & 48 & -80\\ -32 & -23 & 64\\ -8 & 0 & 17\end{pmatrix}$ substituted to give $\mathbf{A}^{-1}=\frac{1}{15}\begin{pmatrix}43 & 78 & -116\\ 4 & 9 & -8\\ 22 & 42 & -59\end{pmatrix}$.
		2	

Q6.

Question	Answer	Marks	Guidance
		M1 A1	Finds characteristic equation.
	$\mathbf{A}^3-3\mathbf{A}^2-\mathbf{A}+3\mathbf{I}=0\Rightarrow\mathbf{A}^4-3\mathbf{A}^3-\mathbf{A}^2+3\mathbf{A}=0$	**M1 A1**	Substitutes **A** and multiplies through by **A**.
	$\mathbf{A}^4=3\left(3\mathbf{A}^2+\mathbf{A}-3\mathbf{I}\right)+\mathbf{A}^2-3\mathbf{A}=10\mathbf{A}^2-9\mathbf{I}$	**M1 A1**	Substitutes $\mathbf{A}^3$.
		6	

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Q7.

Question	Answer	Marks	Guidance
6(a)	$(\lambda-1)(\lambda-2)(\lambda+3)=\lambda^3-7\lambda+6\,[=0]$	**M1 A1**	Finds characteristic equation.
	$\mathbf{P}^2-7\mathbf{I}+6\mathbf{P}^{-1}=0$	**M1**	Substitutes **P** and multiplies through by $\mathbf{P}^{-1}$.
	$\mathbf{P}^2=\begin{pmatrix}1&18&24\\0&4&-6\\0&0&9\end{pmatrix}$ leading to $\mathbf{P}^{-1}=\begin{pmatrix}1&-3&-4\\0&\frac{1}{2}&1\\0&0&-\frac{1}{3}\end{pmatrix}$	**M1 A1**	
		5	

Question	Answer	Marks	Guidance
6(b)	$\mathbf{A}=\mathbf{P}\begin{pmatrix}4&0&0\\0&5&0\\0&0&6\end{pmatrix}\mathbf{P}^{-1}$	**M1**	Applies $\mathbf{A}=\mathbf{PDP}^{-1}$.
	$\begin{pmatrix}4&30&36\\0&10&36\\0&0&-18\end{pmatrix}\begin{pmatrix}1&-3&-\\0&\frac{1}{2}&1\\0&0&-\frac{1}{3}\end{pmatrix}$ or $\begin{pmatrix}1&6&6\\0&2&6\\0&0&-3\end{pmatrix}\begin{pmatrix}4&-12&-16\\0&\frac{5}{2}&5\\0&0&-2\end{pmatrix}$	**M1 A1**	Multiplies two adjacent matrices.
	$\begin{pmatrix}4&3&2\\0&5&-2\\0&0&6\end{pmatrix}$	**A1**	A1 linked to first M1.
		4	
6(c)	Eigenvectors: $\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}6\\2\\0\end{pmatrix},\begin{pmatrix}6\\6\\-3\end{pmatrix}$	**B1**	Accept any non-zero multiples of these.
	Eigenvalues: 64, 125, 216	**B1**	
		2	

Question	Answer	Marks
8(a)	$\begin{vmatrix} 3 & 1 & 1 \\ a & 6 & -1 \\ 0 & a & -2 \end{vmatrix} = 0 \Rightarrow 3(-12+a)-(-2a)+a^2=0$	**M1**
	$a^2+5a-36=0 \Rightarrow a=4,-9$	**M1 A1**
		3
8(b)	$(\lambda-3)(\lambda-6)(\lambda+2)=\lambda^3-7\lambda^2+36=0$	**B1**
	$36\mathbf{I}=7\mathbf{A}^2-\mathbf{A}^3 \Rightarrow 36\left(\mathbf{A}^{-1}\right)^2=7\mathbf{I}-\mathbf{A}$	**M1**
	$\left(\mathbf{A}^{-1}\right)^2=\frac{1}{36}\begin{pmatrix} 4 & -1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 9 \end{pmatrix}$	**M1 A1**
		4

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Question	Answer	Marks
8(c)	Eigenvalues of **A** are $3, 6$ and -2.	**B1**
	$\lambda = 3$: $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 1 \\ 0 & 3 & -1 \end{vmatrix} = \begin{pmatrix} -4 \\ 0 \\ 0 \end{pmatrix} \sim \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$	**M1 A1**
	$\lambda = 6$: $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 1 & 1 \\ 0 & 0 & -1 \end{vmatrix} = \begin{pmatrix} -1 \\ -3 \\ 0 \end{pmatrix}$ $\lambda = -2$: $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 5 & 1 & 1 \\ 0 & 8 & -1 \end{vmatrix} = \begin{pmatrix} -9 \\ 5 \\ 40 \end{pmatrix}$	**A1 A1**
	Thus $\mathbf{P} = \begin{pmatrix} 1 & 1 & -9 \\ 0 & 3 & 5 \\ 0 & 0 & 40 \end{pmatrix}$ and $\mathbf{D} = \begin{pmatrix} 243 & 0 & 0 \\ 0 & 7776 & 0 \\ 0 & 0 & -32 \end{pmatrix}$ or $\mathbf{D} = \begin{pmatrix} 3^5 & 0 & 0 \\ 0 & 6^5 & 0 \\ 0 & 0 & -2^5 \end{pmatrix}$	**M1 A1**
		7

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Q9.

Question	Answer	Marks
3(a)	$\begin{vmatrix} 5-\lambda & -1 & 7 \\ 0 & 6-\lambda & 0 \\ 7 & 7 & 5-\lambda \end{vmatrix} = 0$	**B1**
		M1
	$\lambda = -2, 6, 12$	**A1 A1**
		4

Question	Answer	Marks
3(b)	$-\mathbf{A}^3 + 16\mathbf{A}^2 - 36\mathbf{A} - 144\mathbf{I} = 0$	**B1**
	$144\mathbf{A}^{-1} = -\mathbf{A}^2 + 16\mathbf{A} - 36\mathbf{I}$	**M1**
	$\mathbf{A}^2 = \begin{pmatrix} 74 & 38 & 70 \\ 0 & 36 & 0 \\ 70 & 70 & 74 \end{pmatrix} \Rightarrow \mathbf{A}^{-1} = \frac{1}{24}\begin{pmatrix} -5 & -9 & 7 \\ 0 & 4 & 0 \\ 7 & 7 & -5 \end{pmatrix}$	**M1 A1**
		4

Question	Answer	Marks	Guidance
9(i)	$\mathbf{A}^2\mathbf{e} = \mathbf{A}(\mathbf{Ae}) = \lambda\mathbf{Ae} = \lambda^2\mathbf{e}$	**M1 A1**	AG
		2	
9(ii)	Eigenvalues of $\mathbf{A}$ are $n, 2n$ and $3n$.	**B1**	
	$\lambda = n$: $\mathbf{e_1} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 3 \\ 0 & n & 0 \end{vmatrix} = \begin{pmatrix} -3n \\ 0 \\ 0 \end{pmatrix} = t\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$	**M1 A1**	Uses vector product (or equations) to find corresponding eigenvectors.
	$\lambda = 2n$: $\mathbf{e_2} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -n & 1 & 3 \\ 0 & 0 & n \end{vmatrix} = \begin{pmatrix} n \\ n^2 \\ 0 \end{pmatrix} = t\begin{pmatrix} 1 \\ n \\ 0 \end{pmatrix}$	**A1**	
	$\lambda = 3n$: $\mathbf{e_3} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2n & 1 & 3 \\ 0 & -n & 0 \end{vmatrix} = \begin{pmatrix} 3n \\ 0 \\ 2n^2 \end{pmatrix} = t\begin{pmatrix} 3 \\ 0 \\ 2n \end{pmatrix}$	**A1**	
	Eigenvalues of $\mathbf{A} + n\mathbf{I}$ are $2n, 3n$ and $4n$	**B1**	
	Thus $\mathbf{P} = \begin{pmatrix} 1 & 1 & 3 \\ 0 & n & 0 \\ 0 & 0 & 2n \end{pmatrix}$ and $\mathbf{D} = \begin{pmatrix} (2n)^2 & 0 & 0 \\ 0 & (3n)^2 & 0 \\ 0 & 0 & (4n)^2 \end{pmatrix}$	**M1 A1**	Or correctly matched permutations of columns.
		8	

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Question	Answer	Marks	Guidance
11E(i)	$\mathbf{P}=\begin{pmatrix}1&-1&0\\1&0&1\\0&b&-1\end{pmatrix}$ $\mathbf{D}=\begin{pmatrix}2&0&0\\0&1&0\\0&0&3\end{pmatrix}$	**B1 B1**	Writes **P** and **D** (accept correctly matched permutations of columns)
	$\det\mathbf{P}=-b-1$	**B1**	
	$\mathbf{P}^{-1}=\frac{1}{b+1}\begin{pmatrix}b&-1&-b\\1&1&b\\1&1&-1\end{pmatrix}^{\mathrm{T}}=\frac{1}{b+1}\begin{pmatrix}b&1&1\\-1&1&1\\-b&b&-1\end{pmatrix}$	**M1 A1**	Finds inverse of **P**. (Adj ÷ Det).
	$\mathbf{A}=\mathbf{PDP}^{-1}$	**M1**	Applies $\mathbf{A}=\mathbf{PDP}^{-1}$
	$\mathbf{A}=\frac{1}{b+1}\begin{pmatrix}1&-1&0\\1&0&1\\0&b&-1\end{pmatrix}\begin{pmatrix}2b&2&2\\-1&1&1\\-3b&3b&-3\end{pmatrix}$	**M1 A1**	Multiplies two adjacent matrices. $\mathbf{PD}=\begin{pmatrix}2&-1&0\\2&0&3\\0&b&-3\end{pmatrix}$
	$\frac{1}{b+1}\begin{pmatrix}1+2b&1&1\\-b&3b+2&-1\\2b&-2b&3+b\end{pmatrix}$	**A1**	

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Question	Answer	Marks	Guidance
11E(i)	**Alternative method for 11E(i)**		
	$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{22} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}$ $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{22} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\begin{pmatrix} -1 \\ 0 \\ b \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ b \end{pmatrix}$ $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{22} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ -3 \end{pmatrix}$	**B3**	Matches eigenvalues with eigenvectors to form systems of equations. B1 for each correctly matched pair.
	Three values found correctly from solving systems of equations. Another three values found correctly from solving systems of equations. Attempting to find final three values from systems of equations.	**M1 A1** **M1 A1** **M1**	
	$\mathbf{A} = \frac{1}{b+1}\begin{pmatrix} 1+2b & 1 & 1 \\ -b & 3b+2 & -1 \\ 2b & -2b & 3+b \end{pmatrix}$	**A1**	Final answer given in matrix form.
		9	
11E(ii)	$\mathbf{A}^{-1}\begin{pmatrix} 0 \\ 2 \\ -2 \end{pmatrix} = 2\mathbf{A}^{-1}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \frac{2}{3}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$	**M1 A1**	M1 for $\mathbf{A}^{-1}\begin{pmatrix} 0 \\ 2 \\ -2 \end{pmatrix} = 2\mathbf{A}^{-1}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$.
		2	

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Question	Answer	Marks	Guidance
11E(iii)	$\mathbf{A}^n\begin{pmatrix}1\\1\\0\end{pmatrix}=2^2\begin{pmatrix}1\\1\\0\end{pmatrix}\Rightarrow n=2$	**B1**	Uses $\mathbf{A}^n\begin{pmatrix}1\\1\\0\end{pmatrix}=2^n\begin{pmatrix}1\\1\\0\end{pmatrix}$
	$\mathbf{A}^n\begin{pmatrix}-1\\0\\b\end{pmatrix}=\begin{pmatrix}-1\\0\\b^{-1}\end{pmatrix}\Rightarrow b=b^{-1}\Rightarrow b=1$	**M1 A1**	M1 for stating $b=b^{-1}$ (since $\mathbf{A}^n\begin{pmatrix}-1\\0\\b\end{pmatrix}=\begin{pmatrix}-1\\0\\b\end{pmatrix}$). Must be only one answer ($b=1$) for A1.
		3	

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Q12.

Question	Answer	Marks	Guidance
8(i)	Eigenvalues of (upper diagonal matrix) $\mathbf{A}$ are $2, m$ and 1. (Or from characteristic equation: $(\lambda-2)(\lambda-m)(\lambda-1)=0$)	**B1**	
	$\lambda=2:\ \mathbf{e}_1=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\0&m-2&1\\0&0&-1\end{vmatrix}=\begin{pmatrix}2-m\\0\\0\end{pmatrix}=t\begin{pmatrix}1\\0\\0\end{pmatrix}$	**M1 A1**	Uses vector product (or equations) to find corresponding eigenvectors.
	$\lambda=m:\ \mathbf{e}_2=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\2-m&m&1\\0&0&7\end{vmatrix}=\begin{pmatrix}7m\\7(m-2)\\0\end{pmatrix}=t\begin{pmatrix}m\\m-2\\0\end{pmatrix}$	**A1**	
	$\lambda=1:\ \mathbf{e}_3=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&m&1\\0&m-1&7\end{vmatrix}=\begin{pmatrix}6m+1\\-7\\m-1\end{pmatrix}=t\begin{pmatrix}6m+1\\-7\\m-1\end{pmatrix}$	**A1**	
	Thus $\mathbf{P}=\begin{pmatrix}1&m&6m+1\\0&m-2&-7\\0&0&m-1\end{pmatrix}$ and $\mathbf{D}=\begin{pmatrix}2&0&\\0&m&0\\0&0&1\end{pmatrix}$	**M1 A1 FT**	Or correctly matched permutations of columns. No follow through on two or more zero eigenvectors.
		7	

Question	Answer	Marks	Guidance
8(ii)	$\mathbf{M}^7\mathbf{P}=\mathbf{P}\mathbf{D}^7\mathbf{P}^{-1}\mathbf{P}=\mathbf{P}\mathbf{D}^7=\begin{pmatrix}1&m&6m+1\\0&m-2&-7\\0&0&m-1\end{pmatrix}\begin{pmatrix}2^7&0&0\\0&m^7&0\\0&0&1\end{pmatrix}$	**M1 A1 FT**	Applies $\mathbf{M}^7=\mathbf{P}\mathbf{D}^7\mathbf{P}^{-1}$
	$=\begin{pmatrix}2^7&m^8&6m+1\\0&m^8-2m^7&-7\\0&0&m-1\end{pmatrix}$	**A1**	Order of columns might be swapped depending on **P**.
		3	

Question	Answer	Marks	Guidance
10(i)	$\begin{pmatrix} 1 & 5 & 1 \\ 1 & -2 & -2 \\ 2 & 3 & \theta \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 5 & 1 \\ 0 & -7 & -3 \\ 0 & 0 & \theta+1 \end{pmatrix}$	**M1 A1**	Reduces to echelon form. At least one row operation for M1.
	$r(\mathbf{A}) = 3$ if $\theta \neq -1$	**A1**	
	$r(\mathbf{A}) = 2$ if $\theta = -1$	**B1**	
		4	
10(ii)	$\begin{aligned} x \quad +5y \quad +z &= -1 \\ -7y \quad -3z &= 1 \\ (\theta+1)z &= (\theta+1) \end{aligned}$	**M1**	Uses reduced form of augmented matrix or eliminates variables from scratch.
	$z = 1,\ y = -\frac{4}{7},\ x = \frac{6}{7}$	**A1** **A1**	One correct. All three correct.
		3	
10(iii)	$\begin{aligned} x \quad +5y \quad +z &= -1 \\ -7y \quad -3z &= 1 \\ (\theta+1)z &= (\theta+1) \end{aligned}$ $z = t$	**M1**	Uses parameter.
	$y = -\frac{3t+1}{7},\ x = \frac{8t-2}{7}$	**A1 A1**	
		3	

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Question	Answer	Marks	Guidance
10(iv)	$x+5y+\quad z=-1,$ $-7y\quad -3z=1,$ $(\theta+1)z=\phi+1$	**M1**	Uses reduced form of augmented matrix or eliminates variables from scratch
	$\theta=-1\Rightarrow\phi=-1$ so no solution (inconsistent).	**A1**	
		2	

Q14.

Question	Answer	Marks	Guidance
11O(i)	e.g. $2\mathbf{e}$	**B1**	Allow any scalar multiple $\mu\mathbf{e}$ where $\mu\neq 0,1$
		1	
11O(ii)	Eigenvector: $\mathbf{e}$, Eigenvalue: λ^n	**B1 B1**	Allow any scalar multiple $\mu\mathbf{e}$ where $\mu\neq 0$ Note: $A^n\,\mathbf{e}=\lambda^n\,\mathbf{e}$ SCB1
		2	

Question	Answer	Marks	Guidance
11O(iii)	Eigenvalues of (diagonal matrix) $\mathbf{A}$: $\lambda = 3,7,1$ (Or from characteristic equation: $(\lambda-3)(\lambda-7)(\lambda-1)=0$)	**B1**	
	$\lambda = 3$: $\mathbf{e}_1 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 4 & 0 \\ 4 & 8 & -2 \end{vmatrix} = \begin{pmatrix} -8 \\ 4 \\ 0 \end{pmatrix} = t\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$	**M1 A1**	Uses vector product (or equations) to find corresponding eigenvectors
	$\lambda = 7$: $\mathbf{e}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -4 & 0 & 0 \\ 4 & 8 & -6 \end{vmatrix} = \begin{pmatrix} 0 \\ -24 \\ -32 \end{pmatrix} = t\begin{pmatrix} 0 \\ 3 \\ 4 \end{pmatrix}$	**A1**	
	$\lambda = 1$: $\mathbf{e}_3 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 6 & 0 \\ 4 & 8 & 0 \end{vmatrix} = \begin{pmatrix} 0 \\ 0 \\ -8 \end{pmatrix} = t\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$	**A1**	
	Thus $\mathbf{P} = \begin{pmatrix} -2 & 0 & 0 \\ 1 & 3 & 0 \\ 0 & 4 & 1 \end{pmatrix}$ and $\mathbf{D} = \begin{pmatrix} 3^n & 0 & 0 \\ 0 & 7^n & 0 \\ 0 & 0 & 1 \end{pmatrix}$.	**B1 FT B1 FT**	Or correctly matched permutations of columns. FT on non-zero and distinct eigenvalues and vectors
		7	
11O(iv)	$\sum_{n=1}^{N}\left(k^n A^n - k^{n+1}A^{n+1}\right) = kA - k^{N+1}A^{N+1}$	**M1 A1**	Method of differences
	$k^{N+1}A^{N+1} \to 0$ as $N \to \infty$ for	**M1**	
	$-\frac{1}{7} < k < \frac{1}{7}$.	**A1**	
		4	

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Question	Answer	Marks	Guidance
5(i)	$\mathbf{Ae} = \lambda\mathbf{e}$ **SOI**	**B1**	
	$\mathbf{A}^3\mathbf{e} = \mathbf{A}^2(\mathbf{Ae}) = \lambda\mathbf{A}(\mathbf{Ae}) = \lambda^2(\mathbf{Ae}) = \lambda^3\mathbf{e}$	**M1**	Substitutes for $\mathbf{Ae}$
	So eigenvalue is λ^3. Special case: states eigenvalue is λ^3 B1	**A1**	
5(ii)	Eigenvalues of $\mathbf{A}$ are 2 and 3.	**B1**	
	Eigenvectors of $\mathbf{A}$ are $\begin{pmatrix}1\\1\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$.	**M1**	AEF (allow any non-zero scalar multiple).
	So $\mathbf{P} = \begin{pmatrix}1 & 0\\1 & 1\end{pmatrix}$	**A1**	**Alt method**: Find $\mathbf{A^3 + I} = \begin{pmatrix}9 & 0\\-19 & 28\end{pmatrix}$ B1
	and $\mathbf{D} = \begin{pmatrix}2^3+1 & 0\\0 & 3^3+1\end{pmatrix} = \begin{pmatrix}9 & 0\\0 & 28\end{pmatrix}$ Columns of **P** and **D** can be permuted, but must match.	**M1** **A1FT**	Eigenvalues (9, 28) and vectors $\begin{pmatrix}1\\1\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$ M1, A1. **P** and **D** FT on eigenvalues M1, A1FT
		8	

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Question	Answer	Marks	Guidance
5(i)	$\mathbf{Ae} = \lambda\mathbf{e}$ and $\mathbf{Be} = \mu\mathbf{e} \Rightarrow \mathbf{Ae} + \mathbf{Be} = \lambda\mathbf{e} + \mu\mathbf{e}$	**M1**	Adds equations.
	$\Rightarrow (\mathbf{A} + \mathbf{B})\mathbf{e} = (\lambda + \mu)\mathbf{e}$	**A1**	
		2	
5(ii)	$\begin{pmatrix} 2 & 0 & 1 \\ -1 & 2 & 3 \\ 1 & 0 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \\ 3 \end{pmatrix}$	**M1**	Multiples matrix with vector.
	$\Rightarrow \lambda = 3\mathbb{k}$	**A1**	Finds one eigenvalue.
	$\lambda = 1,\ 2$	**A1**	Finds other two.
		3	
5(iii)	$\mathbf{P} = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 4 & 1 \\ 1 & -1 & 0 \end{pmatrix}$	**B1**	
	$\mathbf{D} = \begin{pmatrix} 7^3 & 0 & 0 \\ 0 & 6^3 & 0 \\ 0 & 0 & 3^3 \end{pmatrix} = \begin{pmatrix} 343 & 0 & 0 \\ 0 & 216 & 0 \\ 0 & 0 & 27 \end{pmatrix}$	**M1 A1**	Or correctly matched permutations of columns.
		3	

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Question	Answer	Marks	Guidance
2(i)	2	**B1**	Stated
		1	
2(ii)	Negative eigenvalue $= -2$	**B1**	Stated
	$\mathbf{A}+2\mathbf{I}=\begin{pmatrix}4&3&1\\0&0&1\\0&0&3\end{pmatrix}$ $\begin{vmatrix}i&j&k\\4&3&1\\0&0&1\end{vmatrix}$	**M1**	**Uses** vector product (or equations) to find corresponding eigenvector.
	$\begin{pmatrix}3\\-4\\0\end{pmatrix}$	**A1**	Accept any non-zero scalar multiple of $\begin{pmatrix}3\\-4\\0\end{pmatrix}$.
		3	
2(iii)	An eigenvalue of $A+A^6$ is $2+2^6=66$, 62 or 2	**B1**	
	Corresponding eigenvector is $\begin{pmatrix}1\\0\\0\end{pmatrix}$ $\begin{pmatrix}3\\-4\\0\end{pmatrix}$ or $\begin{pmatrix}-6\\1\\3\end{pmatrix}$ oe	**B1**	
		2	

Q18.

4(i)	$\begin{vmatrix}1&3&k\\2&-1&-\\1&1&2\end{vmatrix}=0$	**M1 M1**	Using the determinant *Alt method:* Uses row operations
	$\Rightarrow k=8$	**A1**	
	Total:	**3**	

Question	Answer	Marks	Guidance
4(ii)	$x+3y+8z=4$ (1) $4x-2y-10z=-5$ (2)	**M1**	
	From (1) and (2) obtain $y=-3x$ or other correct expression $x+y+2z=1$ (3) Substitute $x=t$ and $y=-3t$ in (3) to obtain z.	**A1 FT**	**Alternative:** Find REF for augmented matrix and form equations (**M1**) Correctly (**A1**)
	$\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\0\\\frac{1}{2}\end{pmatrix}+t\begin{pmatrix}1\\-3\\1\end{pmatrix}$ or $\begin{pmatrix}\frac{1}{-2}\\\frac{3}{2}\\0\end{pmatrix}+t\begin{pmatrix}1\\-3\\1\end{pmatrix}$ (OE)	**A1**	
	Total:	**3**	
Q19. 5(i)	Eigenvectors are **i** + 2**j** + 2**k**, **i** + **k** and **j** + **k** (OE) for $\lambda = 1, -1$ and -2 respectively.	**M1A1**	
	(Award **M1A1** for any one correct and **A1** for each of the other two.)	**A1A1**	
	Total:	**4**	
5(ii)	Eigenvalues for **B** are $-1, -3$ and -4.	**B1**	
	Eigenvectors are the same as for **A** respectively.	**B1 FT**	
	Total:	**2**	

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Question	Answer	Marks	Guidance
10(i)	$\begin{pmatrix} 6 & -8 & 7 \\ 7 & -9 & 7 \\ 6 & -6 & 5 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ -2 \\ 0 \end{pmatrix} \Rightarrow \lambda_1 = -2$	**M1A1**	
	Total:	**2**	
10(ii)	$\lambda_2 = -1 \Rightarrow \mathbf{e}_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$	**M1A1**	OE
	Total:	**2**	
10(iii)	Det $\mathbf{A} = 10 \quad \Rightarrow \lambda_3 = 5 \quad \Rightarrow \mathbf{e}_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$	**B1B1B1**	OE
	Total:	**3**	

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Question	Answer	Marks	Guidance
10(iv)	$\mathbf{P} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix}$ $\mathbf{D} = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 5 \end{pmatrix}$ (Check for consistency.)	**B1FT**	
		B1FT	
	Det $\mathbf{P} = -1$ (From working) Adj $\mathbf{P} = \begin{pmatrix} 1 & -2 & 1 \\ -1 & 1 & 0 \\ -1 & 1 & -1 \end{pmatrix}$ $\Rightarrow \mathbf{P}^{-1} = \begin{pmatrix} -1 & 2 & -1 \\ 1 & -1 & 0 \\ 1 & -1 & 1 \end{pmatrix}$	**M1A1**	Alternative Method: Reduced row echelon form M1, A1
	$\mathbf{A}^n = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} (-2)^n & 0 & 0 \\ 0 & (-1)^n & 0 \\ 0 & 0 & 5^n \end{pmatrix} \begin{pmatrix} -1 & 2 & -1 \\ 1 & -1 & 0 \\ 1 & -1 & 1 \end{pmatrix}$	**M1**	Allow $\mathbf{A}^n = \mathbf{P}\,\mathbf{D}^n\,\mathbf{P}^{-1}$
	$= \ldots = \begin{pmatrix} \left[5^n + (-1)^n - (-2)^n\right] & \left[2.(-2)^n + (-1)^{n+1} - 5^n\right] & \left[5^n - (-2)^n\right] \\ \left[5^n - (-2)^n\right] & \left[2.(-2)^n - 5^n\right] & \left[5^n - (-2)^n\right] \\ \left[5^n - (-1)^n\right] & \left[(-1)^n - 5^n\right] & \left[5^n\right] \end{pmatrix}$	**A1**	
	Total:	**6**	

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Question	Answer	Marks	Guidance
11E(i)	$\mathbf{Ae} = \lambda\mathbf{e}$ and $\mathbf{Be} = \mu\mathbf{e}$	**M1A1**	
	$\mathbf{ABe} = \mathbf{A}\mu\mathbf{e} = \mu\mathbf{Ae} = \mu\lambda\mathbf{e} = \lambda\mu\mathbf{e}$	**M1**	AG
		3	
11E(ii)	$(\lambda+1)(\lambda^2-5\lambda+6)=0$	**A1**	
	$(\lambda+1)(\lambda-2)(\lambda-3)=0$	**A1**	
	$\lambda = -1,2,3.$	**M1**	
	Eigenvectors are $\begin{pmatrix}1\\-1\\0\end{pmatrix}$, $\begin{pmatrix}1\\-1\\1\end{pmatrix}$ and $\begin{pmatrix}1\\0\\1\end{pmatrix}$ respectively.	**A1A1**	Uses either vector product or equations to find eigenvectors
		6	
11E(iii)	$\begin{pmatrix}3&6&1\\1&-2&-1\\6&6&-2\end{pmatrix}\begin{pmatrix}1\\-1\\0\end{pmatrix}=\begin{pmatrix}-3\\3\\0\end{pmatrix}\Rightarrow \mu_1=-3$	**M1**	
	Similarly, other two eigenvalues of **B** are – 2 and 4.	**A1**	
	Eigenvalues of **AB** are 3 , – 4 and 12	**A1**	
	Corresponding eigenvectors are $\begin{pmatrix}1\\-1\\0\end{pmatrix}$, $\begin{pmatrix}1\\-1\\1\end{pmatrix}$ and $\begin{pmatrix}1\\0\\1\end{pmatrix}$.	**A1**	
		4	

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Q22.

Qu	Solution	Part Marks
10	Eigenvalues are:–2, –1, 1 Eigenvectors are; $\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}$ (oe)	**B1** **M1A1** **A1** [4]
	$\mathbf{P}=\begin{pmatrix}1&1&0\\0&1&1\\0&0&1\end{pmatrix}, \mathbf{D}=\begin{pmatrix}-2&0&0\\0&-1&0\\0&0&1\end{pmatrix}$ or equivalent (in correct order)	**B1√B1√**
	$\mathbf{p}^{-1}=\begin{pmatrix}1&-1&1\\0&1&-1\\0&0&1\end{pmatrix}$	**M1A1**
	$\left(\mathbf{P}^{-1}\mathbf{AP}\right)^n=\mathbf{P}^{-1}\mathbf{A}^n\mathbf{P}=\mathbf{D}^n$ $\Rightarrow \mathbf{A}^n=\begin{pmatrix}1&1&0\\0&1&1\\0&0&1\end{pmatrix}\begin{pmatrix}(-2)^n&0&0\\0&(-1)^n&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&-1&1\\0&1&-1\\0&0&1\end{pmatrix}$ Accept $\mathbf{PD}^n\mathbf{P}^{-1}$ here.	**M1A1**
	$\mathbf{A}^n=\begin{pmatrix}1&1&0\\0&1&1\\0&0&1\end{pmatrix}\begin{pmatrix}(-2)^n&-(-2)^n&(-2)^n\\0&(-1)^n&(-1)^{n+1}\\0&0&1\end{pmatrix}$	**M1**
	$=\begin{pmatrix}(-2)^n&-(-2)^n+(-1)^n&(-2)^n+(-1)^{n+1}\\0&(-1)^n&(-1)^{n+1}+1\\0&0&1\end{pmatrix}$	**A1** [8]

Q23.

3	$\begin{vmatrix}k&1&1\\1&k&1\\1&1&k\end{vmatrix}=0 \Rightarrow k^3-3k+2=0$	**M1A1**
	$(k-1)^2(k+2)=0 \Rightarrow k=1,-2$	**M1A1** [4]
	$k=1: \Rightarrow x+y+z=2$ and $x+y+z=-1 \Rightarrow$ inconsistent. $k=-2: \Rightarrow x=z-1$ Hence $(x,y,z)=(\,[-1+t],\,t,\,t\,)$	**B1** **M1** **A1** [3]

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Q24.

Qu	Solution	Marks
11 (e)	$\lambda = 1: \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -3 & -3 \\ -8 & 5 & -3 \end{vmatrix} = \begin{pmatrix} 24 \\ 24 \\ -24 \end{pmatrix} \sim \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ (Or by equations.)	**M1A1**
	$\lambda = 4: \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & -3 & -3 \\ -8 & 2 & -3 \end{vmatrix} = \begin{pmatrix} 15 \\ 15 \\ -30 \end{pmatrix} \sim \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$	**A1** [3]
	$\begin{pmatrix} 1 & -3 & -3 \\ -8 & 6 & -3 \\ 8 & -2 & 7 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 9 \\ -9 \end{pmatrix} \Rightarrow \lambda = 9$	**M1A1** [2]
	$\mathbf{P} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ -1 & -2 & -1 \end{pmatrix}$ $\mathbf{D} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{pmatrix}$	**B1B1**
	Det $\mathbf{P} = 1$	**B1**√
	Adj $\mathbf{P} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & -1 \\ -1 & 1 & 0 \end{pmatrix} \Rightarrow \mathbf{P}^{-1} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & -1 \\ -1 & 1 & 0 \end{pmatrix}$. (Or by row operations)	**M1A1**√ [5]
	$\mathbf{C} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$	**B1**
	$\mathbf{P}^{-1}\mathbf{AP} = \mathbf{C}^2 \Rightarrow \mathrm{A} = \mathbf{PC}^2\mathbf{P}^{-1} = \mathbf{PCP}^{-1}.\ \mathbf{PCP}^{-1}$ (i.e. $\mathbf{B} = \mathbf{PCP}^{-1}$)	**M1**
	$\mathbf{PC} = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 2 & 3 \\ -1 & -4 & -3 \end{pmatrix} \Rightarrow \mathbf{B} = \begin{pmatrix} 1 & -1 & -1 \\ -2 & 2 & -1 \\ 2 & 0 & 3 \end{pmatrix}$ (Intermediate step required.)	**M1A1** [4]
	(Note: **B** may not be unique.)	

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Q25.

3	$\mathbf{P} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$, $\mathbf{D} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ soi	**B1B1**	
	$\mathbf{P}^{-1}\mathbf{AP} = \mathbf{D} \Rightarrow \mathbf{A} = \mathbf{PDP}^{-1}$ soi	**M1**	
	$\mathbf{P}^{-1} = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$	**M1A1**	
	$\mathbf{PD} = \begin{pmatrix} -1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{pmatrix}$ or $\mathbf{DP}^{-1} = \begin{pmatrix} -1 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{pmatrix}$	**M1**	
	$\mathbf{A} = \begin{pmatrix} -1 & 2 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$.	**A1**	[7]
	ALT METHOD: $Av_1 = -v_1$, $Av_2 = v_2$, $Av_3 = 2v_3$ **M1, A1** Multiply out, solve each equation **M1, A1, A1, A1** $\mathbf{A} = \begin{pmatrix} -1 & 2 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$ **B1**√		

Q26.

2	$\begin{vmatrix} 2 & -3 & 4 \\ 3 & -1 & 0 \\ 1 & 2 & k \end{vmatrix} = 0 \Rightarrow 7k = -28 \Rightarrow k = -4$	M1A1 (2)
	Add 1st and 3rd $\Rightarrow 3x - y = 2$ (Same as 2nd) (OE)	M1
	Set $x = t$ (for example) (OE)	M1
	$\Rightarrow y = 3t - 2$, $z = \frac{7}{4}t - \frac{5}{4}$ (OE) – many forms	A1A1 (4)
		Total: 6

Q27.

10

$\begin{pmatrix} 2 & 2 & -3 \\ 2 & 2 & 3 \\ -3 & 3 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ 3 \\ -3 \end{pmatrix} \Rightarrow$ eigenvalue is – 3. M1A1 (2)

$\lambda = 4$: $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 2 & -3 \\ -3 & 3 & -1 \end{vmatrix} = \begin{pmatrix} 7 \\ 7 \\ 0 \end{pmatrix} \sim \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ (Or by equations) M1A1

$\lambda = 6$: $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -4 & 2 & -3 \\ -3 & 3 & -3 \end{vmatrix} = \begin{pmatrix} 3 \\ -3 \\ -6 \end{pmatrix} \sim \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$ A1 (3)

$\mathbf{P} = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & -1 \\ 1 & 0 & -2 \end{pmatrix}$ $\mathbf{D} = \begin{pmatrix} -3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{pmatrix}$ B1√B1√ (2)

M1;A1

$\mathbf{B = QAQ^{-1} = QPAP^{-1}Q^{-1}; = QPA\,(QP)^{-1}}$ B1

$\mathbf{QP} = \begin{pmatrix} -2 & 15 & -17 \\ -1 & 5 & -7 \\ 0 & 3 & -3 \end{pmatrix}$ **(CAO)**

Eigenvalues are –3, 4 and 6 (same as those for **A**). (CAO) B1

Eigenvectors are: $\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 15 \\ 5 \\ 3 \end{pmatrix}, \begin{pmatrix} 17 \\ 7 \\ 3 \end{pmatrix}$. B1 (5)

Total: 12

Q28.

11 O	$\begin{vmatrix} 12 & -4 & 2 \\ -4 & \alpha+9 & 6 \\ 2 & 6 & 7 \end{vmatrix} = 0 \Rightarrow 80\alpha + 80 = 0 \Rightarrow \alpha = -1$	M1A1 A1 (3)
(i)	$\begin{vmatrix} 3-\lambda & -4 & 2 \\ -4 & -1-\lambda & 6 \\ 2 & 6 & -2-\lambda \end{vmatrix} = 0$	M1
	$\Rightarrow \lambda^3 - 63\lambda + 162 = 0$ $\Rightarrow (\lambda+9)(\lambda-3)(\lambda-6) = 0 \Rightarrow \lambda_1 = 6$ and $\lambda_2 = 3$.	M1A1 M1A1 (5)
(ii)	$\lambda = -9 \Rightarrow \mathbf{e} = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$ (Award M1A1 for any one $\lambda_1 = 6 \Rightarrow \mathbf{e}_1 = \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix}$ and $\lambda_2 = 3 \Rightarrow \mathbf{e}_2 = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$ and A1 for the other two).	M1A1 A1 (3)
	$\mathbf{Mx} = \mathbf{M}(a\mathbf{e}_1 + b\mathbf{e}_2) = a\mathbf{Me}_1 + b\mathbf{Me}_2$ $= a\lambda_1\mathbf{e}_1 + b\lambda_2\mathbf{e}_2$ $= 6a\mathbf{e}_1 + 3b\mathbf{e}_2$	B1 B1 B1 (3) **Total 14**

Q29.

6	$\lambda = 1: \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 10 & -8 & 10 \\ 7 & -5 & 7 \end{vmatrix} = \begin{pmatrix} -6 \\ 0 \\ 6 \end{pmatrix} \sim \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ oe	**M1A1**
	$\lambda = 3: \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 0 & 0 \\ 10 & -10 & 10 \end{vmatrix} = \begin{pmatrix} 0 \\ 20 \\ 20 \end{pmatrix} \sim \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ oe	**A1** [3]
	$\begin{pmatrix} 1 & 0 & 0 \\ 10 & -7 & 10 \\ 7 & -5 & 8 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -4 \\ -2 \end{pmatrix} = -2 \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} \Rightarrow \lambda = -2$	**M1A1** [2]
	$\mathbf{D} = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$, $\mathbf{P} = \begin{pmatrix} 0 & 1 & 0 \\ 2 & 0 & 1 \\ 1 & -1 & 1 \end{pmatrix}$ (or other multiples or permutations).	**B1√ B1√**
	Det $\mathbf{P} = -1$ (or 1 depending on permutation).	**B1**
	Adj $\mathbf{P} = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 0 & 0 \\ -2 & 1 & 2 \end{pmatrix} \Rightarrow \mathbf{P}^{-1} = \begin{pmatrix} -1 & 1 & -1 \\ 1 & 0 & 0 \\ 2 & -1 & 2 \end{pmatrix}$ (or other permutations).	**M1A1** [5] **Total 10**

Q30.

9	$\mathbf{M}\begin{pmatrix}0\\1\\-1\end{pmatrix}=\begin{pmatrix}0\\-1\\1\end{pmatrix}\Rightarrow$ eigenvalue $=-1\ \ (=\lambda_1)$	M1A1 [2]
	$\lambda_1+\lambda_2+\lambda_3=-8\Rightarrow\lambda_2+\lambda_3=-7$	M1
	$-\lambda_2\lambda_3=\begin{vmatrix}-2&2&2\\2&1&2\\-3&-6&-7\end{vmatrix}=-10+16-18=-12\Rightarrow\lambda_2\lambda_3=12$	M1A1 A1A1
	$\lambda_2=-3$ and $\lambda_3=-4$ or vice versa. (S.C. Award M1A1 for $\lambda^3+8\lambda^2+19\lambda+12=0$ and A1 for both $\lambda_2=-3$ and $\lambda_3=-4$.)	
	$\lambda_2=-3\Rightarrow\mathbf{e}_2=\begin{pmatrix}2\\-1\\0\end{pmatrix}$ (OE)	M1A1
	$\lambda_3=-4\Rightarrow\mathbf{e}_3=\begin{pmatrix}1\\0\\-1\end{pmatrix}$ (OE)	A1 [8]

Q31.

8	$\mathbf{Ae} = \lambda\mathbf{e} \Rightarrow \mathbf{A}^{-1}\mathbf{Ae} = \mathbf{A}^{-1}\lambda\mathbf{e}$ $\therefore \mathbf{e} = \mathbf{A}^{-1}\lambda\mathbf{e} = \lambda\mathbf{A}^{-1}\mathbf{e} \Rightarrow \frac{\mathbf{1}}{\lambda}\mathbf{e} = \mathbf{A}^{-1}\mathbf{e}$ $\mathbf{Ae} + \mathbf{A}^{-1}\mathbf{e} = \lambda\mathbf{e} + \frac{1}{\lambda}\mathbf{e} \Rightarrow (\mathbf{A} + \mathbf{A}^{-1})\mathbf{e} = \left(\lambda + \frac{1}{\lambda}\right)\mathbf{e}$ $\begin{vmatrix} \mathrm{i} & \mathrm{j} & \mathrm{k} \\ 1 & 0 & 1 \\ -1 & 1 & 3 \end{vmatrix} = \begin{pmatrix} -1 \\ -4 \\ 1 \end{pmatrix}$ (OE) $\begin{pmatrix} 2 & 0 & 1 \\ -1 & 2 & 3 \\ 1 & 0 & 2 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} \Rightarrow \lambda = 2$ $\begin{pmatrix} 2 & 0 & 1 \\ -1 & 2 & 3 \\ 1 & 0 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \\ 3 \end{pmatrix} \Rightarrow \lambda = 3$ (S.C. Award B1 for eigenvalues obtained from characteristic equation and not matched to eigenvectors.) $\mathbf{P} = \begin{pmatrix} -1 & 0 & 1 \\ -4 & 1 & 2 \\ 1 & 0 & 1 \end{pmatrix}$ (OE) (F.T. on their calculated eigenvector.) Eigenvalues are $\left(1+\frac{1}{1}\right)^3 = 8$, $\left(2+\frac{1}{2}\right)^3 = \frac{125}{8}$, $\left(3+\frac{1}{3}\right)^3 = \frac{1000}{27}$ $\mathbf{D} = \begin{pmatrix} 8 & 0 & 0 \\ 0 & \frac{125}{8} & 0 \\ 0 & 0 & \frac{1000}{27} \end{pmatrix}$ (F.T. requires decent attempt at $\left(\lambda + \frac{1}{\lambda}\right)^3$.)	M1 A1 [2] B1 [1] M1A1 [2] B1 B1 [2] B1√ M1A1 B1√ [4]

Q32.

5	$\begin{vmatrix} 1 & -1 & 2 \\ 1 & -1 & -3 \\ 1 & -1 & 7 \end{vmatrix} = 0$	M1
	$\Rightarrow 5a + 5 = 0 \Rightarrow a = -1$	A1A1 (3)
	$\begin{pmatrix} 1 & -1 & 2 & \mid & 4 \\ 1 & -1 & -3 & \mid & -5 \\ 1 & -1 & 7 & \mid & 13 \end{pmatrix} \rightarrow ... \rightarrow \begin{pmatrix} 1 & -1 & 2 & \mid & 4 \\ 0 & 0 & 5 & \mid & 9 \\ 0 & 0 & 0 & \mid & 0 \end{pmatrix}$, or by elimination methods,	M1
	$x - y + 2z = 4$ $5z = 9$	A1
	$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0.4 \\ 0 \\ 1.8 \end{pmatrix} + t\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ (OE)	M1A1 (4)
	Planes form a prism, or words to that effect.	B1 (1) **[8]**

Q33.

11O (i)	$\mathbf{Ae} = \lambda\mathbf{e}$; since $\mathbf{A}$ is non-singular $\mathbf{Ae} \neq 0 \Rightarrow \lambda \neq 0$ ($\mathbf{e} \neq 0$) .	M1A1 (2)
(ii)	$\mathbf{Ae} = \lambda\mathbf{e} \Rightarrow \mathbf{A}^{-1}\mathbf{Ae} = \mathbf{A}^{-1}\lambda\mathbf{e}$ $\Rightarrow \mathbf{e} = \lambda\mathbf{A}^{-1}\mathbf{e} \Rightarrow \mathbf{A}^{-1}\mathbf{e} = \frac{1}{\lambda}\mathbf{e}$	M1 A1 (2)
	Eigenvalues of **A** are -2 , -1 , 3 (1 mark for any one, 2 marks	B2,1,0
	Corresponding eigenvectors are: $\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}2\\1\\0\end{pmatrix}, \begin{pmatrix}-6\\25\\20\end{pmatrix}$ (M1A1 for one, A1 for each other.) (N.B. May come from using eigenvalues of $\mathbf{A} + 3\mathbf{I}$.)	M1A1 A1 A1
	$\mathbf{P} = \begin{pmatrix}1 & 2 & -6\\0 & 1 & 25\\0 & 0 & 20\end{pmatrix}$	B1√
	Eigenvalues of $\mathbf{A} + 3\mathbf{I}$ are 1 , 2 , 6 (Award B3 if obtained from $\mathbf{A} + 3\mathbf{I}$.)	B1√
	Eigenvalues of $(\mathbf{A} + 3\mathbf{I})^{-1}$ are $1, \frac{1}{2}, \frac{1}{6}$	B1√
	$\mathbf{D} = \begin{pmatrix}1 & 0 & 0\\0 & \frac{1}{2} & 0\\0 & 0 & \frac{1}{6}\end{pmatrix}$ (CAO)	B1 (10) **[14]**

DIFFERENTIATION

5 The variables x and y are such that $y = 0$ when $x = 0$ and

$$(x+1)y + (x+y+1)^3 = 1.$$

(a) Show that $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{3}{4}$ when $x = 0$. [3]

(b) Find the Maclaurin's series for y up to and including the term in x^2. [7]

Q2.

2 **(a)** Find the coefficient of x^2 in the Maclaurin's series for $-\ln\cos x$. [4]

(b) Find the length of the arc of the curve with equation $y = -\ln\cos x$ from the point where $x = 0$ to the point where $x = \frac{1}{4}\pi$. [4]

Q3.

4 It is given that

$$x = -t + \tan^{-1} t \qquad \text{and} \qquad y = t + \sinh^{-1} t .$$

(a) Show that $\dfrac{\mathrm{d}y}{\mathrm{d}x} = -\dfrac{t^2 + 1 + \sqrt{t^2 + 1}}{t^2}$. [4]

(b) Find the value of $\frac{d^2y}{dx^2}$ when $t = \frac{3}{4}$. [5]

Q4.

7 **(a)** It is given that $y = \text{sech}^{-1}\left(x+\frac{1}{2}\right)$.

Express cosh y in terms of x and hence show that $\sinh y\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{\left(x+\frac{1}{2}\right)^2}$. [3]

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(b) Find the first three terms in the Maclaurin's series for $\text{sech}^{-1}\left(x+\frac{1}{2}\right)$ in the form

$$\ln a + bx + cx^2,$$

where a, b and c are constants to be determined. [7]

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Q5.

2 Find the Maclaurin's series for $\ln\cosh x$ up to and including the term in x^4. [7]

Q6.

6 **(a)** Starting from the definitions of sinh and cosh in terms of exponentials, prove that

$$2\sinh^2 x = \cosh 2x - 1.$$ [3]

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(b) Find the solution to the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + y\coth x = 4\sinh x$$

for which $y = 1$ when $x = \ln 3$. [7]

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Q7.

3 The curve C has equation

$$xy^3 - 4x^3y = 3.$$

(a) Show that, at the point $(-1, 1)$ on C, $\frac{\mathrm{d}y}{\mathrm{d}x} = 11$. [3]

(b) Find the value of $\frac{d^2y}{dx^2}$ at the point $(-1, 1)$. [5]

Q8.

1 Find the Maclaurin's series for $e^x \tan x$ from first principles up to and including the term in x^2. [5]

Q9.

1 It is given that $y = \sinh(x^2) + \cosh(x^2)$.

(a) Use standard results from the list of formulae (MF19) to find the Maclaurin's series for y in terms of x up to and including the term in x^4. [2]

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(b) Deduce the value of $\frac{\mathrm{d}^4 y}{\mathrm{d}x^4}$ when $x = 0$. [1]

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(c) Use your answer to part **(a)** to find an approximation to $\int_0^{\frac{1}{2}} y \, \mathrm{d}x$, giving your answer as a rational fraction in its lowest terms. [2]

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Q10.

5 The curve C has parametric equations

$$x = 3t + 2t^{-1} + at^3, \quad y = 4t - \tfrac{3}{2}t^{-1} + bt^3, \quad \text{for } 1 \leqslant t \leqslant 2,$$

where a and b are constants.

(a) It is given that $a = \frac{2}{3}$ and $b = -\frac{1}{2}$.

Show that $\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 = \frac{25}{4}(t^2 + t^{-2})^2$ and find the exact length of C. [6]

(b) It is given instead that $a = b = 0$.

Find the value of $\frac{d^2y}{dx^2}$ when $t = 1$. [4]

Q11.

2 It is given that $y = 2^x$.

(a) By differentiating $\ln y$ with respect to x, show that $\frac{\mathrm{d}y}{\mathrm{d}x} = 2^x \ln 2$. [3]

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(b) Write down $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$. [1]

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(c) Hence find the first three terms in the Maclaurin's series for 2^x. [3]

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Q12.

6 **(a)** Starting from the definitions of tanh and sech in terms of exponentials, prove that

$$1-\tanh^2\theta = \operatorname{sech}^2\theta. \qquad [3]$$

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The variables x and y are such that $\tanh y = \cos\left(x+\frac{1}{4}\pi\right)$, for $-\frac{1}{4}\pi < x < \frac{3}{4}\pi$.

(b) By differentiating the equation $\tanh y = \cos\left(x+\frac{1}{4}\pi\right)$ with respect to x, show that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\operatorname{cosec}\left(x+\frac{1}{4}\pi\right). \qquad [4]$$

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(c) Hence find the first three terms in the Maclaurin's series for $\tanh^{-1}\left(\cos\left(x+\frac{1}{4}\pi\right)\right)$ in the form $\frac{1}{2}\ln a+bx+cx^2$, giving the exact values of the constants a, b and c. [5]

Q13.

1 A curve C has equation $\cos y = x$, for $-\pi < x < \pi$.

(i) Use implicit differentiation to show that

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -\cot y\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2. \qquad [4]$$

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(ii) Hence find the exact value of $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ at the point $\left(\frac{1}{2}, \frac{1}{3}\pi\right)$ on C. [2]

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Q14.

The positive variables y and t are related by

$$y = a^t,$$

where a is a positive constant.

(i) (a) By differentiating $\ln y$ with respect to t, show that $\dfrac{\mathrm{d}y}{\mathrm{d}t} = a^t \ln a$. [3]

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(b) Write down $\dfrac{\mathrm{d}^2y}{\mathrm{d}t^2}$. [1]

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(ii) Determine the set of values of a for which the infinite series

$$y + \frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\mathrm{d}^2y}{\mathrm{d}t^2} + \frac{\mathrm{d}^3y}{\mathrm{d}t^3} + \ldots$$

is convergent. [3]

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A curve has parametric equations

$$x = t^a, \quad y = a^t.$$

(iii) Find $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$ in terms of a and t, and show that, when $t = 2$,

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 2^{1-2a}(1 - a + 2\ln a)\ln a. \qquad [7]$$

Q15.

1 The variables x and y are such that $y = -1$ when $x = 0$ and

$$\left(x + \frac{\mathrm{d}y}{\mathrm{d}x}\right)^3 = y^2 + x.$$

(i) Find the value of $\frac{\mathrm{d}y}{\mathrm{d}x}$ when $x = 0$. [1]

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(ii) Find also the value of $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ when $x = 0$. [4]

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Q16.

11 The curve C has equation

$$x^2 + 2xy = y^3 - 2.$$

(i) Show that $A\,(-1,\ 1)$ is the only point on C with x-coordinate equal to -1. [2]

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For $n \geqslant 1$, let A_n denote the value of $\dfrac{\mathrm{d}^n y}{\mathrm{d}x^n}$ at the point $A\,(-1,\ 1)$.

(ii) Show that $A_1 = 0$. [3]

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(iii) Show that $A_2 = \frac{2}{5}$. [3]

Let $I_n = \int_{-1}^{0} x^n \frac{\mathrm{d}^n y}{\mathrm{d}x^n} \, \mathrm{d}x$.

(iv) Show that for $n \geqslant 2$,

$$I_n = (-1)^{n+1} A_{n-1} - nI_{n-1}.$$ [3]

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(v) Deduce the value of I_3 in terms of I_1. [2]

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Q17.

3 A curve C has equation $\tan y = x$, for $x > 0$.

(i) Use implicit differentiation to show that

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -2x\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2. \qquad [3]$$

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(ii) Hence find the value of $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ at the point $\left(1, \frac{1}{4}\pi\right)$ on C. [2]

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[Turn over

Q18.

4 A curve C has equation $x^3 - 3xy + y^2 = 4$. Find the value of $\frac{d^2y}{dx^2}$ at the point (0, 2) of C. [7]

Q19.

5 The curve C has equation $2x^3 + 3x^2y - 3y^3 - 16 = 0$.

(i) Find the coordinates of the point A on C at which $\frac{dy}{dx} = 0$ and $x \neq 0$. [5]

(ii) Find the value of $\frac{d^2y}{dx^2}$ at A. [3]

Q20.

4 The curve C has equation $y = -\ln(1 - x^2)$ for $-\frac{1}{2} \leqslant x \leqslant \frac{1}{2}$. Show that

$$1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = \left(\frac{1 + x^2}{1 - x^2}\right)^2. \qquad [2]$$

Show further that $\dfrac{1 + x^2}{1 - x^2}$ may be expressed in the form $\dfrac{P}{1 + x} + \dfrac{Q}{1 - x} + R$, where P, Q and R are constants to be determined. [2]

Find the exact arc length of C. [4]

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[Turn over

Q21.

8 A curve C has equation $x^2 + 4xy - y^2 + 20 = 0$. Show that, at stationary points on C, $x = -2y$. [3]

Find the coordinates of the stationary points on C, and determine their nature by considering the value of $\frac{d^2y}{dx^2}$ at the stationary points. [8]

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[Turn over

Q22.

6 A curve has equation $x^2 - 6xy + 25y^2 = 16$. Show that $\frac{dy}{dx} = 0$ at the point (3, 1). [4]

By finding the value of $\frac{d^2y}{dx^2}$ at the point (3, 1), determine the nature of this turning point. [5]

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Q23.

7 The curve C has equation $x^2 + 2xy - 4y^2 + 20 = 0$. Show that if the tangent to C at the point (x, y) is parallel to the x-axis then $x + y = 0$. [3]

Hence find the coordinates of the stationary points on C, and determine their nature. [7]

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Q24.

1 The curve C is defined parametrically by

$$x = 2\cos^3 t \quad \text{and} \quad y = 2\sin^3 t, \quad \text{for } 0 < t < \tfrac{1}{2}\pi.$$

Show that, at the point with parameter t,

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \tfrac{1}{6}\sec^4 t \operatorname{cosec} t. \qquad [4]$$

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Q25.

7 The curve C has parametric equations

$$x = \sin t, \quad y = \sin 2t, \quad \text{for } 0 \leqslant t \leqslant \pi.$$

Find $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$ in terms of t. [5]

Hence, or otherwise, find the coordinates of the stationary points on C and determine their nature. [5]

DIFFERENTITATION
MS

Q1.

PUBLISHED

Question	Answer	Marks	Guidance
5(a)	$\frac{d}{dx}((x+1)y)=(x+1)y'+y$	B1	Differentiates $(x+1)y$ correctly.
	$\frac{d}{dx}\left((x+y+1)^3\right)=3(x+y+1)^2(y'+1)$	B1	Differentiates $(x+y+1)^3$ correctly.
	$(1)y'+3(y'+1)=0 \Rightarrow y'=-\frac{3}{4}$	B1	Substitutes $(0,0)$, AG.
		3	
5(b)	$(x+1)y''+2y'$	B1	Differentiates $(x+1)y'+y$.
	$+3(x+y+1)^2y''+6(1+y')^2(x+y+1)=0$	B1 B1	Differentiates $3(x+y+1)^2(1+y')$. Expect $6(x+y+1)(1+y')+6(x+y+1)(1+y')y'$ instead of $6(1+y')^2(x+y+1)$.
	$y''+2\left(-\frac{3}{4}\right)+3y''+6\left(\frac{1}{4}\right)^2=0$	M1	Substitutes $(0,0)$.
	$y''=\frac{9}{32}$	A1	
	$y=y(0)+y'(0)x+\frac{y''(0)}{2!}x^2=-\frac{3}{4}x+\frac{9}{64}x^2$	M1 A1	Finds Maclaurin's series.
		7	

Q2.

Question	Answer	Marks	Guidance
2(a)	$\frac{d}{dx}(-\ln\cos x)=\tan x$	B1	Finds first derivative.
	$\frac{d}{dx}(\tan x)=\sec^2 x$	B1	Finds second derivative.
	$\frac{y''(0)}{2!}=\frac{1}{2}$	M1 A1	Evaluates second derivative at zero. Accept $\frac{1}{2}x^2$.
		4	

Question	Answer	Marks	Guidance
2(b)	$\int_0^{\frac{1}{4}\pi}\sqrt{1+\tan^2 x}\,dx$	M1	Forms correct integral
	$=\int_0^{\frac{1}{4}\pi}\sec x\,dx$	A1	
	$=\left[\ln\left(\tan\left(\frac{1}{2}x+\frac{1}{4}\pi\right)\right)\right]_0^{\frac{1}{4}\pi}$	M1	Integrates (answer given in List MF19). Accept $\ln(\sec x+\tan x)$.
	$\ln\tan\left(\frac{3}{8}\pi\right)=0.881$	A1	Accept $\ln\left(\sqrt{2}+1\right)$.
		4	

Q3.

Question	Answer	Marks	Guidance
4(a)	$\frac{dy}{dt}=1+\frac{1}{\sqrt{1+t^2}}$	**B1**	
	$\frac{dx}{dt}=-1+\frac{1}{1+t^2}$	**B1**	
	$\frac{dy}{dx}=\frac{dy}{dt}\times\frac{dt}{dx}=\frac{\sqrt{1+t^2}+1}{\sqrt{1+t^2}}\times\frac{1+t^2}{-t^2}$	**M1**	Uses chain rule.
	$-\sqrt{1+t^2}\left(\frac{\sqrt{1+t^2}+1}{t^2}\right)=-\frac{1+t^2+\sqrt{1+t^2}}{t^2}$	**A1**	AG
		4	

Question	Answer	Marks	Guidance
4(b)	$\frac{d}{dt}\left(\frac{dy}{dx}\right)=-\frac{t^2\left(2t+t\left(1+t^2\right)^{-\frac{1}{2}}\right)-2t\left(1+t^2+\sqrt{1+t^2}\right)}{t^4}$	**M1 A1**	Applies product or quotient rule.
	$\frac{d^2y}{dx^2}=\frac{t^2\left(1+t^2\right)^{-\frac{1}{2}}-2\left(1+\sqrt{1+t^2}\right)}{t^5}\times\left(1+t^2\right)$ $=-\frac{\left(t^2+2+2\sqrt{t^2+1}\right)\sqrt{1+t^2}}{t^5}$	**M1 A1**	Uses chain rule. Simplification not required for A1.
	$=-\frac{80}{3}$ when $t=\frac{3}{4}$.	**A1**	
		5	

Q4.

Question	Answer	Marks	Guidance
7(a)	$\operatorname{sech} y = \dfrac{1}{\cosh y} = x + \frac{1}{2} \Rightarrow \cosh y = \left(x + \frac{1}{2}\right)^{-1}$	**B1**	Relates to $\cosh y$,
	$\sinh y \dfrac{\mathrm{d}y}{\mathrm{d}x}$	**B1**	Differentiates LHS.
	$-\left(x + \frac{1}{2}\right)^{-2}$	**B1**	Differentiates RHS. AG.
		3	
7(b)	$\sinh \dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} + \cosh y \left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = 2\left(x + \dfrac{1}{2}\right)^{-3}$	**M1 A1**	M1 A1 for LHS. B1 for RHS.
		B1	
	$y(0) = \operatorname{sech}^{-1}\left(\frac{1}{2}\right) = \cosh^{-1}(2) = \ln\left(2 + \sqrt{3}\right)$	**M1 A1**	Relates to $\cosh^{-1}$ and uses logarithmic form.
	$y'(0) = -\dfrac{4}{\sqrt{3}} \quad y''(0) = \dfrac{16}{3\sqrt{3}}$	**M1**	Evaluates derivatives at $x = 0$.
	$y = \ln\left(2 + \sqrt{3}\right) - \dfrac{4}{\sqrt{3}}x + \dfrac{8}{3\sqrt{3}}x^2$	**A1**	
	Alternative method for question 7(b)		
	$\dfrac{\mathrm{d}y}{\mathrm{d}x} = -\dfrac{1}{\left(x + \frac{1}{2}\right)^2 \sqrt{\left(x + \frac{1}{2}\right)^{-2} - 1}} = -\dfrac{1}{\left(x + \frac{1}{2}\right)\sqrt{\frac{3}{4} - x - x^2}}$	**B1**	
	$\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{1}{2}\left(x + \frac{1}{2}\right)^{-1}\left(\frac{3}{4} - x - x^2\right)^{-\frac{3}{2}}(-1 - 2x) + \left(\frac{3}{4} - x - x^2\right)^{-\frac{1}{2}}\left(x + \frac{1}{2}\right)^{-2}$	**M1 A1**	
	$y(0) = \operatorname{sech}^{-1}\left(\frac{1}{2}\right) = \cosh^{-1}(2) = \ln\left(2 + \sqrt{3}\right)$	**M1 A1**	Relates to $\cosh^{-1}$ and uses logarithmic form.

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Question	Answer	Marks	Guidance
7(b)	$y'(0) = -\frac{4}{\sqrt{3}} \quad y''(0) = \frac{16}{3\sqrt{3}}$	**M1**	Evaluates derivatives at $x = 0$.
	$y = \ln\left(2+\sqrt{3}\right) - \frac{4}{\sqrt{3}}x + \frac{8}{3\sqrt{3}}x^2$	**A1**	
		7	

Q5.

Question	Answer	Marks	Guidance
2	$\frac{dy}{dx} = \tanh x$	**B1**	Finds first derivative.
	$\frac{d^2y}{dx^2} = \operatorname{sech}^2 x$	**B1**	Finds second derivative.
	$\frac{d^3y}{dx^3} = -2\tanh x \operatorname{sech}^2 x = -2\frac{\sinh x}{\cosh^3 x}$	**M1 A1**	Finds third derivative.
	$\frac{d^4y}{dx^4} = -2\frac{\cosh^2(x) - 3\sinh^2(x)}{\cosh^4(x)} = -2\operatorname{sech}^4 x + 4\tanh^2 x \operatorname{sech}^2 x$	**B1**	Finds fourth derivative. Alternative: $6\tanh^2 x \operatorname{sech}^2 x - 2\operatorname{sech}^2 x.$
	$y(0) = y^{(1)}(0) = y^{(3)}(0) = 0 \quad y^{(2)}(0) = 1 \quad y^{(4)}(0) = -2$	**M1**	Evaluates derivatives at $x = 0$.
	$y = \frac{1}{2}x^2 - \frac{1}{12}x^4$	**A1**	CWO
		7	

Q6.

Question	Answer	Marks	Guidance
6(a)	$\sinh x = \frac{1}{2}\left(e^{x} - e^{-x}\right) \quad \cosh x = \frac{1}{2}\left(e^{x} + e^{-x}\right)$	**B1**	Writes in exponential from.
	$\frac{1}{2}\left(e^{x} - e^{-x}\right)^2 = \frac{1}{2}\left(e^{2x} - 2 + e^{-2x}\right) = \frac{1}{2}\left(e^{2x} + e^{-2x}\right) - 1$	**M1**	Expands.
	$\cosh 2x - 1$	**A1**	AG.
		3	
6(b)	$e^{\int \coth x} = e^{\ln \sinh x} = \sinh x$	**M1 A1**	Finds integrating factor.
	$\frac{d}{dx}(y \sinh x) = 4\sinh^2 x$	**M1**	Correct form on both sides.
	$y \sinh x = 2\left(\frac{1}{2}\sinh 2x - x\right) + C = \sinh 2x - 2x + C$	**M1 A1**	Integrates $\sinh^2 x$ using identity.
	$\sinh \ln 3 = \sinh \ln 9 - 2\ln 3 + C$ leading to $\frac{4}{3} = \frac{40}{9} - 2\ln 3 + C$	**M1**	Substitutes initial conditions into their solution.
	$y \sinh x = \sinh 2x - 2x + 2\ln 3 - \frac{28}{9}$	**A1**	
		7	

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Q7.

Question	Answer	Marks	Guidance
3(a)	$3xy^2y'+y^3$	**B1**	Differentiates xy^3.
	$-4\left(x^3y'+3x^2y\right)=0$	**B1**	Differentiates x^3y.
	$3(-1)(1)y'+1^3-4\left((-1)^3y'+3(-1)^2(1)\right)=0$ leading to $y'=11$	**B1**	Substitutes (–1,1), AG. $y'\left(3xy^2-4x^3\right)=12x^2y-y^3$
		3	

Question	Answer	Marks	Guidance
3(b)	$\left(3xy^2-4x^3\right)y''+y'\left(6xyy'+3y^2-12x^2\right)$	**B1 B1**	Differentiates $\left(3xy^2-4x^3\right)y'$.
	$+3y^2y'-12\left(x^2y'+2xy\right)=0$	**B1**	Differentiates y^3-12x^2y.
	$3xy^2y''-4x^3y''+6xy\left(y'\right)^2+6y^2y'-24x^2y'-24xy=0$ $y''=\dfrac{\left(3xy^2-4x^3\right)\left(12x^2y'+24xy-3y^2y'\right)-\left(12x^2y-y^3\right)\left(6xyy'+3y^2-12x^2\right)}{\left(3xy^2-4x^3\right)^2}$		
	$y''+11(-75)+3(11)-12(9)=0$	**M1**	Substitutes (–1, 1)
	$y''=900$	**A1**	
		5	

Q8.

Question	Answer	Marks	Guidance
1	$e^x\left(\tan x+\sec^2 x\right)$	**B1**	Finds first derivative.
	$e^x\left(\tan x+2\sec^2 x+2\sec^2 x\tan x\right)$	**M1 A1**	Finds second derivative.
	$y(0)=0,\quad y'(0)=1,\quad y''(0)=2$	**M1**	Evaluates derivatives at $x=0$.
	$y=x+x^2$	**A1**	
		5	

Question	Answer	Marks	Guidance
1(a)	$\sinh\left(x^2\right)+\cosh\left(x^2\right)$	**M1**	Combines using correct power series. Or sensible attempt at all four derivatives of y.
	$1+x^2+\frac{1}{2}x^4$	**A1**	
		2	
1(b)	$\frac{1}{2}\times 4!=12$	**B1**	
		1	
1(c)	$\int_0^{\frac{1}{2}} 1+x^2+\frac{1}{2}x^4 \, \mathrm{d}x$	**M1**	Substitutes their power series, must be at least $a+bx^2$.
	$=\left[x+\frac{1}{3}x^3+\frac{1}{10}x^5\right]_0^{\frac{1}{2}}=\frac{523}{960}$	**A1**	
		2	

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Q10.

Question	Answer	Marks	Guidance
5(a)	$\dot{x} = 3 - 2t^{-2} + 2t^2 \qquad \dot{y} = 4 + \frac{3}{2}t^{-2} - \frac{3}{2}t^2$	**B1**	Differentiates x and y with respect to t.
	$4t^4 + 12t^2 + \frac{4}{t^4} - \frac{12}{t^2} + 1 + \frac{23}{2} + \frac{9t^4}{4} + \frac{12}{t^2} + \frac{9}{4t^4} - 12t^2$	**M1**	Expands $\dot{x}^2 + \dot{y}^2$. Accept $25 + \frac{25}{4}\left(t^{-2} - t^2\right)^2$.
	$\frac{25}{4}t^4 + \frac{25}{4}t^{-4} + \frac{25}{2} = \frac{25}{4}\left(t^2 + t^{-2}\right)^2$	**M1 A1**	Factorises $\dot{x}^2 + \dot{y}^2$. AG.
	$\frac{5}{2}\int_1^2 t^2 + t^{-2}\,\mathrm{d}t = \frac{5}{2}\left[\frac{1}{3}t^3 - t^{-1}\right]_1^2 = \frac{85}{12}$	**M1 A1**	Applies correct formula for arc length.
		6	

Question	Answer	Marks	Guidance
5(b)	$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\dot{y}}{\dot{x}} = \frac{4 + \frac{3}{2}t^{-2}}{3 - 2t^{-2}}$	**B1**	Finds $\frac{\mathrm{d}y}{\mathrm{d}x}$.
	$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{4 + \frac{3}{2}t^{-2}}{3 - 2t^{-2}}\right) = \frac{\left(3 - 2t^{-2}\right)\left(-3t^{-3}\right) - \left(4 + \frac{3}{2}t^{-2}\right)\left(4t^{-3}\right)}{\left(3 - 2t^{-2}\right)^2}$	**B1**	Differentiates $\frac{\mathrm{d}y}{\mathrm{d}x}$ with respect to t.
	$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{4 + \frac{3}{2}t^{-2}}{3 - 2t^{-2}}\right) \times \frac{\mathrm{d}t}{\mathrm{d}x} = \frac{-25t^{-3}}{\left(3 - 2t^{-2}\right)^3} = -25$ when $t = 1$.	**M1 A1**	Applies chain rule.
		4	

Q11.

Question	Answer	Marks
2(a)	$\ln y = x\ln 2 \Rightarrow \frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = \ln 2$	**M1 A1**
	$\frac{\mathrm{d}y}{\mathrm{d}x} = y\ln 2 = 2^x \ln 2$ **AG**	**A1**
		3
2(b)	$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 2^x (\ln 2)^2$	**B1**
		1

Question	Answer	Marks
2(c)	$y(0) = 1, \quad y'(0) = \ln 2, \quad y''(0) = (\ln 2)^2$	**B1**
	$2^x = y(0) + y'(0)x + \frac{y''(0)}{2!}x^2 + \ldots = 1 + (\ln 2)x + \frac{(\ln 2)^2}{2}x^2 + \ldots$	**M1 A1**
		3

Q12.

Question	Answer	Marks
6(a)	$\tanh\theta = \dfrac{e^{\theta}-e^{-\theta}}{e^{\theta}+e^{-\theta}} \quad \operatorname{sech}\theta = \dfrac{2}{e^{\theta}+e^{-\theta}}$	**B1**
	$1-\left(\dfrac{e^{\theta}-e^{-\theta}}{e^{\theta}+e^{-\theta}}\right)^2 = \dfrac{\left(e^{\theta}+e^{-\theta}\right)^2-\left(e^{\theta}-e^{-\theta}\right)^2}{\left(e^{\theta}+e^{-\theta}\right)^2} = \dfrac{4}{\left(e^{\theta}+e^{-\theta}\right)^2}$	**M1**
		A1
		3

Question	Answer	Marks
6(b)	$\tanh y = \cos(x+\frac{1}{4}\pi) \quad \Rightarrow \quad \operatorname{sech}^2 y\dfrac{dy}{dx} = -\sin(x+\frac{1}{4}\pi)$	**M1 A1**
	$\left(1-\cos^2(x+\frac{1}{4}\pi)\right)\dfrac{dy}{dx} = -\sin(x+\frac{1}{4}\pi)$	**M1**
	$\dfrac{dy}{dx} = -\dfrac{\sin(x+\frac{1}{4}\pi)}{\sin^2(x+\frac{1}{4}\pi)} = -\operatorname{cosec}(x+\frac{1}{4}\pi)$	**A1**
		4
6(c)	$\dfrac{d^2y}{dx^2} = \cot(x+\frac{1}{4}\pi)\operatorname{cosec}(x+\frac{1}{4}\pi)$	**B1**
	$y'(0) = -\operatorname{cosec}(\frac{1}{4}\pi) \quad y''(0) = \cot(\frac{1}{4}\pi)\operatorname{cosec}(\frac{1}{4}\pi)$	**M1**
	$y(0) = \tanh^{-1}\left(\frac{1}{2}\sqrt{2}\right) = \frac{1}{2}\ln\left(\dfrac{2+\sqrt{2}}{2-\sqrt{2}}\right)$	**M1**
	$y = \frac{1}{2}\ln(3+2\sqrt{2}) - x\sqrt{2} + \frac{1}{2}x^2\sqrt{2}$	**M1 A1**
		5

Question	Answer	Marks	Guidance
1(i)	$-\sin y\dfrac{\mathrm{d}y}{\mathrm{d}x}=1\Rightarrow\dfrac{\mathrm{d}y}{\mathrm{d}x}=-(\sin y)^{-1}$	**M1 A1**	Differentiates implicitly once.
	$\dfrac{\mathrm{d}^2y}{\mathrm{d}^2x}=(\sin y)^{-2}\cos y\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)=-\cot y\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)^2$	**M1 A1**	Differentiates again, AG.
		4	
1(ii)	$\dfrac{\mathrm{d}y}{\mathrm{d}x}=-\left(\sin\dfrac{\pi}{3}\right)^{-1}=-\dfrac{2}{\sqrt{3}}$	**B1**	
	$\dfrac{\mathrm{d}^2y}{\mathrm{d}^2x}=-\dfrac{1}{\sqrt{3}}\left(-\dfrac{2}{\sqrt{3}}\right)^2=-\dfrac{4}{3\sqrt{3}}=-\dfrac{4}{9}\sqrt{3}$	**B1**	AEF, must be exact.
		2	

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Question	Answer	Marks	Guidance
11O(i)(a)	$\ln y = t \ln a$ $\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}t} = \ln a$	**M1 A1**	Differentiates both sides.
	$\frac{\mathrm{d}y}{\mathrm{d}t} = y \ln a = a^t \ln a$	**A1**	AG
		3	
11O(i)(b)	$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} = a^t \left(\ln a\right)^2$	**B1**	
11O(ii)	$r = \ln a$	**B1**	States common ratio.
	$-1 < \ln a < 1 \Rightarrow \mathrm{e}^{-1} < a < \mathrm{e}$	**M1 A1**	Uses convergence condition for a geometric series.
		3	

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Question	Answer	Marks	Guidance
11O(iii)	$\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{\mathrm{d}y}{\mathrm{d}t}\times\dfrac{\mathrm{d}t}{\mathrm{d}x}=\dfrac{a^t\ln a}{at^{a-1}}=\left(\dfrac{\ln a}{a}\right)\left(\dfrac{a^t}{t^{a-1}}\right)$	**M1 A1**	Uses chain rule to find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$
	$\dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)=\left(\dfrac{\ln a}{a}\right)\left(\dfrac{t^{a-1}a^t\ln a-(a-1)t^{a-2}a^t}{t^{2(a-1)}}\right)$	**M1 A1**	Uses quotient or product rule to find $\dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)$
	$\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}=\left(\dfrac{\ln a}{a^2}\right)\left(\dfrac{t^{a-1}a^t\ln a-(a-1)t^{a-2}a^t}{t^{3(a-1)}}\right)$ $=\left(\dfrac{\ln a}{a^2}\right)\left(\dfrac{\ln a-(a-1)t^{-1}}{t^{2(a-1)}}\right)a^t$	**M1 A1**	Uses chain rule to find $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$
	$t=2\Rightarrow\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}=2^{1-2a}\ln a(2\ln a-a+1)$	**A1**	Substitutes $t=2$, AG.
		7	

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Q15.

Question	Answer	Marks	Guidance
1(i)	$\left(0+\frac{\mathrm{d}y}{\mathrm{d}x}\right)^3=(-1)^2+0\Rightarrow\frac{\mathrm{d}y}{\mathrm{d}x}=1$	**B1**	
1(ii)	$3\left(x+\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\left(1+\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)$	**M1 A1**	Method mark for good attempt at implicit differentiation of LHS.
	$=2y\frac{\mathrm{d}y}{\mathrm{d}x}+1$	**B1**	Note: may expand bracket before differentiation but M1 is still for implicit differentiation
	$\Rightarrow 3\left(1+\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)=-2+1\Rightarrow\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=-\frac{4}{3}$	**A1**	
		5	

Q16.

11O(i)	$x=-1\Rightarrow y^3+2y-3=0\Rightarrow(y-1)\left(y^2+y+3\right)=0$	**M1**	Considers cubic polynomial in y.
	There is only one real root ($=1$).	**A1**	
		2	

Question	Answer	Marks	Guidance
11O(ii)	$2x+2\left(x\dfrac{\mathrm{d}y}{\mathrm{d}x}+y\right)$	**B1**	Differentiates LHS correctly.
	$=3y^2\dfrac{\mathrm{d}y}{\mathrm{d}x}$	**B1**	Differentiates RHS correctly.
	$\Rightarrow\left(3y^2-2x\right)\dfrac{\mathrm{d}y}{\mathrm{d}x}=2x+2y$ $x=-1, y=1\Rightarrow\dfrac{\mathrm{d}y}{\mathrm{d}x}=0.$	**B1**	Substitutes $(-1,1)$, AG.
		3	
11O(iii)	$\left(3y^2-2x\right)\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}+\dfrac{\mathrm{d}y}{\mathrm{d}x}\left(6y\dfrac{\mathrm{d}y}{\mathrm{d}x}-2\right)=2+2\dfrac{\mathrm{d}y}{\mathrm{d}x}$	**M1 M1**	Differentiates equation again.
	$x=-1, y=1, \dfrac{\mathrm{d}y}{\mathrm{d}x}=0\Rightarrow\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}=\dfrac{2}{5}$	**A1**	Substitutes $(-1,1)$ and $\dfrac{\mathrm{d}y}{\mathrm{d}x}=0$, AG.
		3	
11O(iv)	$\displaystyle\int_{-1}^{0}x^n\frac{\mathrm{d}^ny}{\mathrm{d}x^n}\mathrm{d}x=\left[x^n\frac{\mathrm{d}^{n-1}y}{\mathrm{d}x^{n-1}}\right]_{-1}^{0}-n\int_{-1}^{0}x^{n-1}\frac{\mathrm{d}^{n-1}y}{\mathrm{d}x^{n-1}}\mathrm{d}x$	**M1 A1**	Integrates by parts.
	$=(-1)^{n+1}A_{n-1}-nI_{n-1}.$	**A1**	AG.
		3	

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Question	Answer	Marks	Guidance
11O(v)	$I_3 = (-1)^4 A_2 - 3I_2 = A_2 - 3(-A_1 - 2I_1) = \frac{2}{5} + 6I_1$	**M1 A1**	Uses reduction formulae.
		2	

Q17.

Question	Answer	Marks	Guidance
3(i)	$\left(1+x^2\right)\frac{\mathrm{d}y}{\mathrm{d}x} = 1$ **or** $\sec^2 y\frac{dy}{dx} = 1 \Rightarrow \left(1+x^2\right)\frac{\mathrm{d}y}{\mathrm{d}x} = 1$ **or** $\frac{dy}{dx} = \cos^2 y$	**M1**	Using implicit differentiation
	$\Rightarrow 2x\frac{\mathrm{d}y}{\mathrm{d}x} + \left(1+x^2\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0 \Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -2x\left(\frac{dy}{dx}\right)^2$ (AG)	**M1 A1**	M1 for good attempt at product rule
	Alt method: $\frac{\mathrm{d}y}{\mathrm{d}x} = \cos^2 y \Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \cos y(-\sin y)\frac{\mathrm{d}y}{\mathrm{d}x} = -2x\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2$	**M1 A1**	M1 for good attempt at implicit differentiation
	Total:	**3**	
3(ii)	$y'(1) = \cos^2\left(\frac{\pi}{4}\right) \Rightarrow y'(1) = \frac{1}{2}$	**B1**	
	$\Rightarrow y''(1) = -\frac{1}{2}$	**B1 FT**	
	Total:	**2**	

Question	Answer	Marks	Guidance
4	$3x^2 - \left(3y + 3x\frac{dy}{dx}\right) + 2y\frac{dy}{dx} = 0$	**B1B1**	SOI
	At (0,2): $-6 + 4\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{3}{2}$	**B1**	
	$6x - 3\frac{dy}{dx} - \left(3\frac{dy}{dx} + 3x\frac{d^2y}{dx^2}\right)$	**M1A1**	
	$+2\left(\frac{dy}{dx}\right)^2 + 2y\frac{d^2y}{dx^2} = 0$	**A1**	
	At (0,2): $-\frac{9}{2} - \frac{9}{2} + \frac{9}{2} + 4\frac{d^2y}{dx^2} = 0 \Rightarrow \frac{d^2y}{dx^2} = \frac{9}{8}$	**A1**	
	Total:	**7**	

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Q19.

Question	Answer	Marks	Guidance
5(i)	$6x^2 + 6xy + 3x^2y' - 9y^2y' = 0 \ (*) \Rightarrow 2x(x+y) = (3y^2 - x^2)y'$	**M1A1**	
	$y' = 0 \text{ and } x \neq 0 \Rightarrow x = -y$	**M1A1**	
	$\Rightarrow 2x^3 - 3x^3 + 3x^3 = 16 \Rightarrow A$ is (2, –2)	**A1**	
		5	
5(ii)	$12x + 6xy' + 6y + 6xy' + 3x^2y'' - \left[18y\left(y'\right)^2 + 9y^2y''\right] = 0$	***M1**	
	$x = 2 \quad y = -2 \quad y' = 0 \Rightarrow 8 - 4 + 4y'' - 12y'' = 0$	**DM1**	
	$\Rightarrow y'' = \frac{1}{2}$	**A1**	
		3	

Q20.

Qu	Soluti	Marks
4	$\frac{dy}{dx} = \frac{2x}{1-x^2}$	**B1**
	$1+\left(\frac{dy}{dx}\right)^2 = 1+\frac{4x^2}{\left(1-x^2\right)^2} = \frac{1-2x^2+x^4+4x^2}{\left(1-x^2\right)^2} = \frac{\left(1+x^2\right)^2}{\left(1-x^2\right)^2}$ **(AG)**	**B1** [2]
	$\frac{1+x^2}{1-x^2} \equiv \frac{1}{1+x}+\frac{1}{1-x}-1$	**M1A1** [2]
	$s = 2\int_0^{\frac{1}{2}}\left(\frac{1}{1+x}+\frac{1}{1-x}-1\right)dx$ (oe)	**M1**
	$= 2\left[\ln\left(\frac{1+x}{1-x}\right)-x\right]_0^{\frac{1}{2}}$	**M1A1**
	$= 2\ln 3 - 1$ (AEF)	**A1** [4]

Q21.

8	$2x+4\left(xy'+y\right)-2yy' = 0 \ \ (*)$	**M1A1**	
	$y' = 0 \Rightarrow 2x+4y = 0 \Rightarrow x = -2y$ (AG)	**A1**	[3]
	At stationary points $4y^2-8y^2-y^2+20 = 0 \Rightarrow 5y^2 = 20 \Rightarrow y = \pm 2$	**M1A1**	
	Coordinates of stationary points are (4 , –2) and (–4,2)	**A1**	
	From $(*)$: $x+2\left(xy'+y\right)-yy' = 0$		
	Differentiating: $1+2\left(xy''+y'+y'\right)-\left(yy''+\left[y'\right]^2\right) = 0$ (or by quotient rule)	**M1A1**	
	At (4 , –2) with $y' = 0$: $1+8y''+2y'' = 0 \Rightarrow y'' = -\frac{1}{10} \Rightarrow$ maximum.	**M1A1**	
	At (–4 , 2) with $y' = 0$: $1-8y''-2y'' = 0 \Rightarrow y'' = \frac{1}{10} \Rightarrow$ minimum.	**A1**	[8]

Q22.

6	$2x - 6\left(x\frac{dy}{dx} + y\right) + 50y\frac{dy}{dx} = 0$ At (3,1) $6 - 18y' - 6 + 50y' = 0 \Rightarrow y' = 0$ (AG)	M1A1 A1 A1 (4)
	$2 - 6(xy'' + y' + y') + 50[(y')^2 + yy''] = 0$	B1B1
	At (3,1) $2 - 18y'' + 50y'' = 0 \Rightarrow y'' = -\frac{1}{16}$	M1A1
	$\Rightarrow$ maximum. (All previous marks required for final mark, i.e. CSO)	A1 (5) **Total: 9**

Q23.

7	$2x + 2\left(x\frac{dy}{dx} + y\right) - 8y\frac{dy}{dx} = 0$ (1^{st} B1 for 1^{st} and 3^{rd} terms, 2^{nd} B1 for rest including = 0).	B1B1
	$\Rightarrow \frac{dy}{dx}(x - 4y) = -(x + y) \Rightarrow y = -x$ when $\frac{dy}{dx} = 0$	B1 (3)
	Substituting for $y \Rightarrow x^2 - 2x^2 - 4x^2 + 20 = 0 \Rightarrow 5x^2 = 20 \Rightarrow x \pm 2$	M1
	Coordinates are (2, –2) and (–2, 2)	A1
	$\frac{d^2y}{dx^2}(x - 4y) + \frac{dy}{dx}\left(1 - 4\frac{dy}{dx}\right) = -1 - \frac{dy}{dx}$ (OE)	M1A1
	Substituting either (2, –2) or (–2, 2) **and** $\frac{dy}{dx} = 0$	dM1
	At (2, –2) $y''(2) = -\frac{1}{10} \Rightarrow$ max.	A1
	At (–2, 2) $y''(2) = \frac{1}{10} \Rightarrow$ min.	A1 (7) **Total 10**

Q24.

1	$\dot{x} = -6\cos^2 t \sin t$, $\dot{y} = 6\sin^2 t \cos t$	**B1**
	$\Rightarrow \dfrac{dy}{dx} = -\tan t$ (OE)	**B1**
	$\dfrac{d^2y}{dx^2} = -\sec^2 t \times \dfrac{-1}{6\cos^2 t \sin t} = \dfrac{1}{6}\sec^4 t \operatorname{cosec} t$ AG	**M1A1** [4] **Total 4**

Q25.

7	$\dot{x} = \cos t$, $\dot{y} = 2\cos 2t \Rightarrow y' = \dfrac{2\cos 2t}{\cos t}$	M1A1
	$y'' = \dfrac{-4\cos t \sin 2t + 2\cos 2t \sin t}{\cos^2 t} \times \dfrac{1}{\cos t} = -\dfrac{4\sin 2t}{\cos^2 t} + \dfrac{2\cos 2t \sin t}{\cos^3 t}$ (OE)	M1A1
	e. $y'' = \dfrac{4\sin^3 t - 6\sin t}{\cos^3 t}$	A1 [5]
	y' or $\dot{y} = 0 \Rightarrow \cos 2t = 0$	M1
	$2t = \dfrac{\pi}{2}$, $\dfrac{3\pi}{2} \Rightarrow t = \dfrac{\pi}{4}$, $\dfrac{3\pi}{4}$	A1
	Stationary points are $\left(\dfrac{1}{\sqrt{2}}, 1\right)$, $\left(\dfrac{1}{\sqrt{2}}, -1\right)$	A1
	(S.C. If only one value of t is given with correct corresponding coordinates – A1)	
	$y''\left(\dfrac{\pi}{4}\right) = -8 \Rightarrow$ max. (CWO)	B1
	$y''\left(\dfrac{3\pi}{4}\right) = +8 \Rightarrow$ min. (CWO)	B1 [5]

INTEGRATION

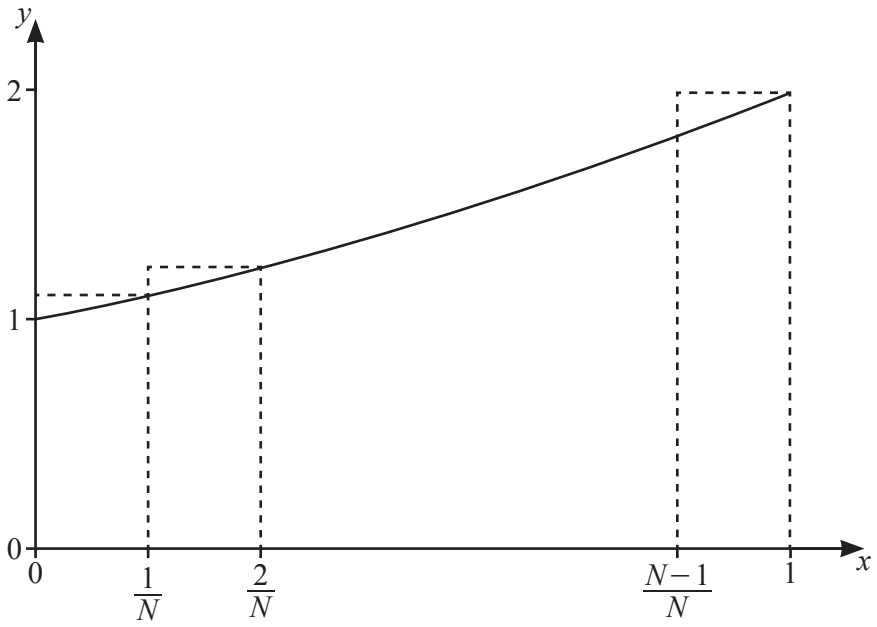
Q1.

1 The curve C has polar equation $r = e^{\frac{3}{4}\theta}$ for $0 \leqslant \theta \leqslant \alpha$.

Given that the length of C is s, find α in terms of s. [5]

Q2.

4 The diagram shows the curve with equation $y = 2^x$ for $0 \leqslant x \leqslant 1$, together with a set of N rectangles each of width $\frac{1}{N}$.



(a) By considering the sum of the areas of these rectangles, show that $\int_0^1 2^x \, \mathrm{d}x < U_N$, where

$$U_N = \frac{2^{\frac{1}{N}}}{N\left(2^{\frac{1}{N}} - 1\right)}.$$ [4]

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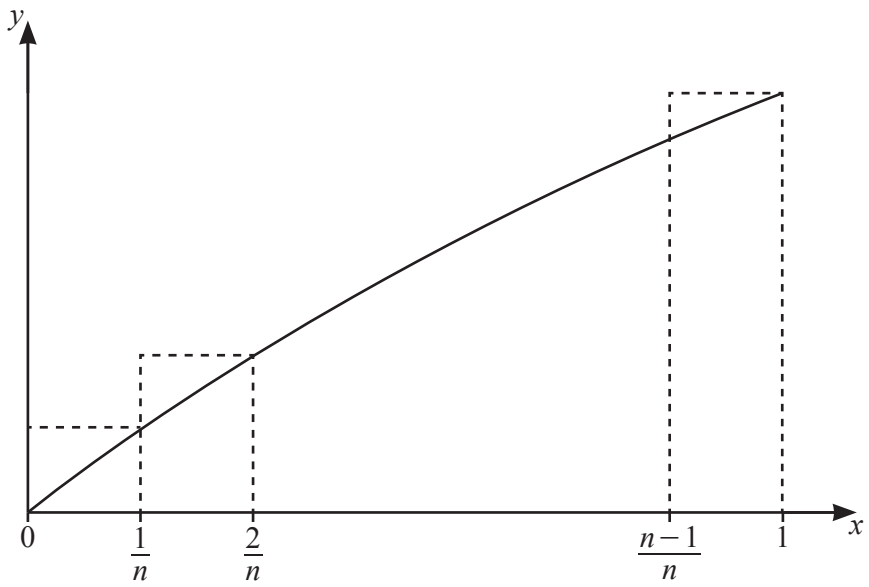
(b) Use a similar method to find, in terms of N, a lower bound L_N for $\int_0^1 2^x \, \mathrm{d}x$. [4]

(c) Find the least value of N such that $U_N - L_N < 10^{-4}$. [2]

Q3.

6



The diagram shows the curve with equation $y = \ln(1+x)$ for $0 \leqslant x \leqslant 1$, together with a set of n rectangles each of width $\frac{1}{n}$.

(a) By considering the sum of the areas of these rectangles, show that $\int_0^1 \ln(1+x)\,\mathrm{d}x < U_n$, where

$$U_n = \frac{1}{n}\ln\frac{(2n)!}{n!} - \ln n.$$ [4]

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(b) Use a similar method to find, in terms of n, a lower bound L_n for $\int_0^1 \ln(1+x)\,\mathrm{d}x$. [4]

(c) By simplifying $U_n - L_n$, show that $\lim_{n\to\infty}(U_n - L_n) = 0$. [2]

Q4.

8 **(a)** Find $\int \sin\theta \cos^n\theta \, \mathrm{d}\theta$, where $n \neq -1$. [2]

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Let $I_{m,n} = \int_0^{\frac{1}{2}\pi} \sin^m\theta \cos^n\theta \, \mathrm{d}\theta$.

(b) Show that, for $m \geqslant 2$ and $n \geqslant 0$,

$$I_{m,n} = \frac{m-1}{m+n} I_{m-2,n}.$$ [5]

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(c) By considering the binomial expansion of $\left(z+\frac{1}{z}\right)^5$, where $z=\cos\theta+\mathrm{i}\sin\theta$, use de Moivre's theorem to show that

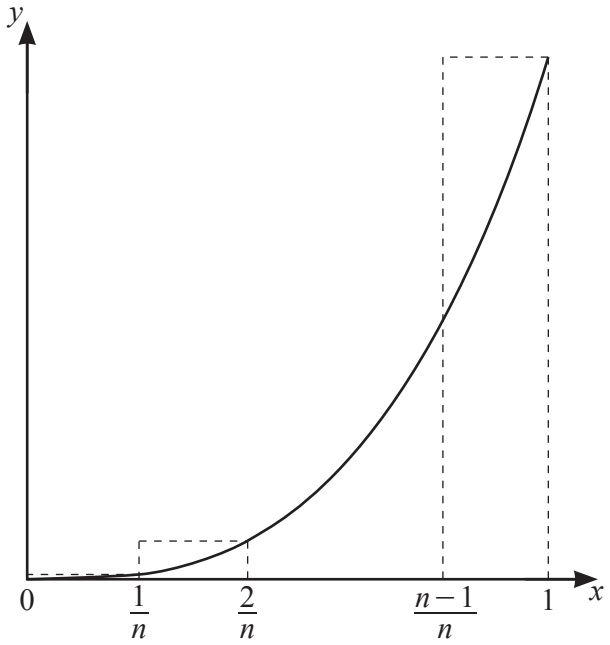
$$\cos^5\theta = a\cos 5\theta + b\cos 3\theta + c\cos\theta,$$

where a, b and c are constants to be determined. [5]

(b) Deduce the exact value of x for which $y = 0$. Give your answer in logarithmic form. [3]

Q5.

3



The diagram shows the curve with equation $y = x^3$ for $0 \leqslant x \leqslant 1$, together with a set of n rectangles of width $\frac{1}{n}$.

(a) By considering the sum of the areas of these rectangles, show that $\int_0^1 x^3 \mathrm{d}x < U_n$, where

$$U_n = \left(\frac{n+1}{2n}\right)^2.$$ [4]

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(b) Use a similar method to find, in terms of n, a lower bound L_n for $\int_0^1 x^3 \mathrm{d}x$. [4]

(c) Find the least value of n such that $U_n - L_n < 10^{-3}$. [2]

Q6.

8 The curve C has parametric equations

$$x = 2\cosh t, \quad y = \tfrac{3}{2}t - \tfrac{1}{4}\sinh 2t, \quad \text{for } 0 \leqslant t \leqslant 1.$$

(a) Find $\dfrac{\mathrm{d}x}{\mathrm{d}t}$ and show that $\dfrac{\mathrm{d}y}{\mathrm{d}t} = 1 - \sinh^2 t$. [3]

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The area of the surface generated when C is rotated through 2π radians about the x-axis is denoted by A.

(b) (i) Show that $A = \pi \int_0^1 \left(\tfrac{3}{2}t - \tfrac{1}{4}\sinh 2t\right)(1 + \cosh 2t)\,\mathrm{d}t$. [4]

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(ii) Hence find A in terms of π, $\sinh 2$ and $\cosh 2$. [6]

Q7.

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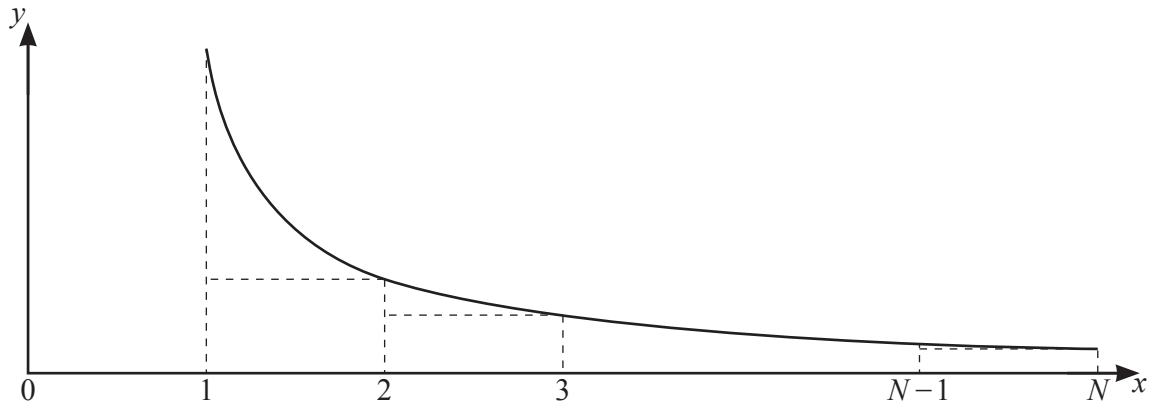

The diagram shows the curve $y = \dfrac{x}{2x^2 - 1}$ for $x \geqslant 1$, together with a set of $N - 1$ rectangles of unit width.

(a) By considering the sum of the areas of these rectangles, show that

$$\sum_{r=1}^{N} \frac{r}{2r^2 - 1} < \tfrac{1}{4}\ln(2N^2 - 1) + 1. \qquad [7]$$

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(b) Use a similar method to find, in terms of N, a lower bound for $\sum_{r=1}^{N} \frac{r}{2r^2-1}$. [3]

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Q8.

7 The integral I_n, where n is an integer, is defined by $I_n = \int_0^{\frac{3}{2}} \left(4+x^2\right)^{-\frac{1}{2}n} \mathrm{d}x$.

(a) Find the exact value of I_1, expressing your answer in logarithmic form. [3]

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(b) By considering $\frac{\mathrm{d}}{\mathrm{d}x}\left(x\left(4+x^2\right)^{-\frac{1}{2}n}\right)$, or otherwise, show that

$$4nI_{n+2} = \tfrac{3}{2}\left(\tfrac{2}{5}\right)^n + (n-1)I_n.$$ [5]

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(c) Find the value of I_5. [3]

Q9.

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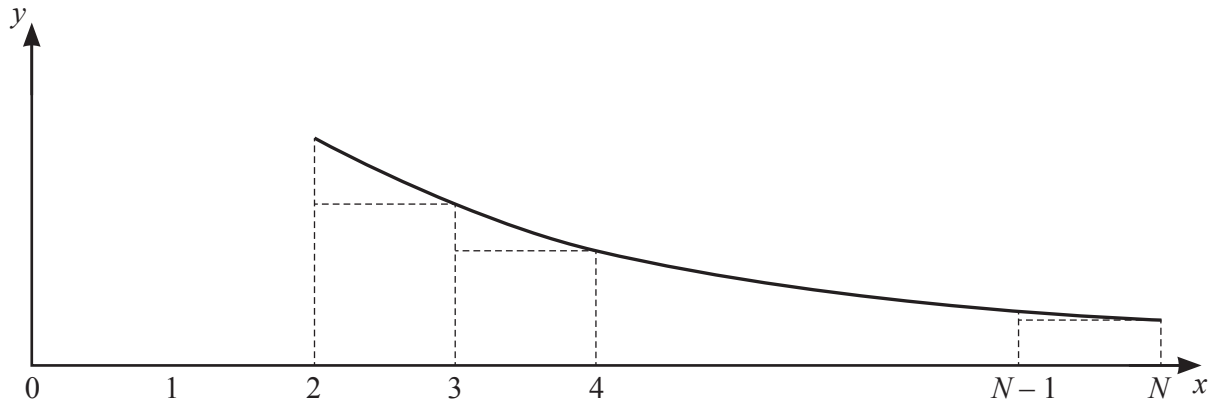

The diagram shows the curve with equation $y = \dfrac{\ln x}{x^2}$ for $x \geqslant 2$, together with a set of $(N-2)$ rectangles of unit width.

(a) By considering the sum of the areas of these rectangles, show that

$$\sum_{r=1}^{N} \frac{\ln r}{r^2} < \frac{2+3\ln 2}{4} - \frac{1+\ln N}{N}. \qquad [7]$$

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(b) Use a similar method to find, in terms of N, a lower bound for $\sum_{r=1}^{N} \frac{\ln r}{r^2}$. [3]

8 **(a)** Starting from the definition of cosh in terms of exponentials, prove that

$$2\cosh^2 A = \cosh 2A + 1. \qquad [3]$$

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The curve C has parametric equations

$$x = 2\cosh 2t + 3t, \quad y = \tfrac{3}{2}\cosh 2t - 4t, \quad \text{for } -\tfrac{1}{2} \leqslant t \leqslant \tfrac{1}{2}.$$

The area of the surface generated when C is rotated through 2π radians about the y-axis is denoted by A.

(b) **(i)** Show that $A = 10\pi \int_{-\frac{1}{2}}^{\frac{1}{2}} (2\cosh 2t + 3t)\cosh 2t \, \mathrm{d}t$. [4]

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(ii) Hence find A in terms of π and e. [7]

Q11.

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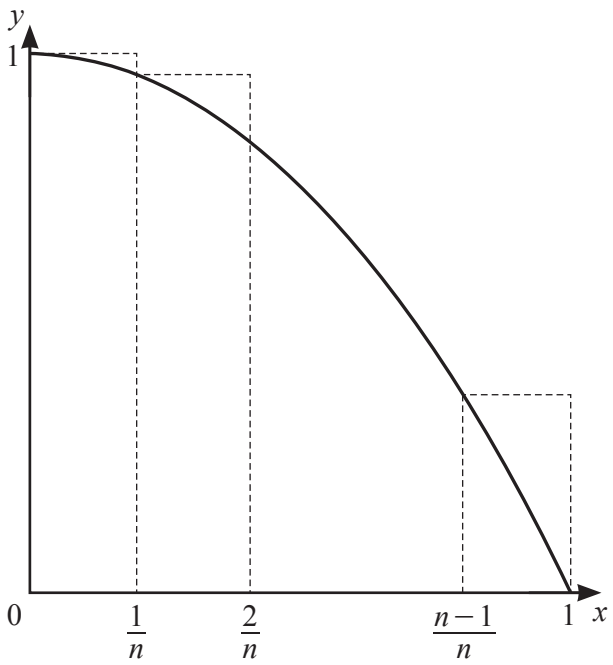

The diagram shows the curve with equation $y = 1 - x^2$ for $0 \leqslant x \leqslant 1$, together with a set of n rectangles of width $\frac{1}{n}$.

(a) By considering the sum of the areas of the rectangles, show that

$$\int_0^1 (1 - x^2)\,\mathrm{d}x < \frac{4n^2 + 3n - 1}{6n^2}. \qquad [4]$$

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(b) Use a similar method to find, in terms of n, a lower bound for $\int_0^1 (1-x^2)\,\mathrm{d}x$. [4]

Q12.

8 **(a)** Starting from the definitions of tanh and sech in terms of exponentials, prove that

$$1-\tanh^2 x = \operatorname{sech}^2 x. \qquad [3]$$

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(b) Using the substitution $u = \tanh x$, or otherwise, find $\int \operatorname{sech}^2 x \tanh^2 x \, \mathrm{d}x$. [2]

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It is given that, for $n \geqslant 0$, $I_n = \int_0^{\ln 3} \operatorname{sech}^n x \tanh^2 x \, \mathrm{d}x$.

(c) Show that, for $n \geqslant 2$,

$$(n+1)I_n = \left(\tfrac{4}{5}\right)^3 \left(\tfrac{3}{5}\right)^{n-2} + (n-2)I_{n-2}. \qquad [5]$$

[You may use the result that $\frac{\mathrm{d}}{\mathrm{d}x}(\operatorname{sech} x) = -\tanh x \operatorname{sech} x$.]

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(d) Find the value of I_4. [3]

Q13.

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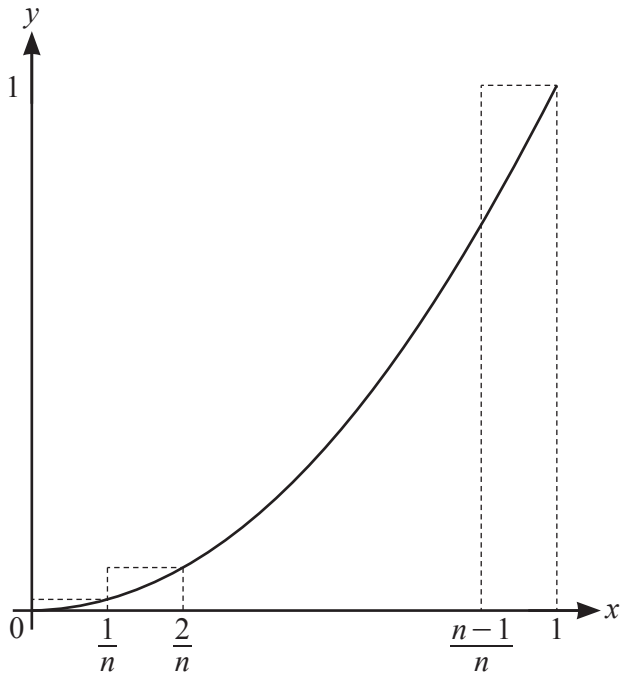

The diagram shows the curve with equation $y = x^2$ for $0 \leqslant x \leqslant 1$, together with a set of n rectangles of width $\frac{1}{n}$.

(a) By considering the sum of the areas of these rectangles, show that

$$\int_0^1 x^2 \, \mathrm{d}x < \frac{2n^2 + 3n + 1}{6n^2}. \qquad [4]$$

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(b) Use a similar method to find, in terms of n, a lower bound for $\int_0^1 x^2 \, \mathrm{d}x$. [4]

Q14.

5 The curves $C_1 : y = \cosh x$ and $C_2 : y = \sinh 2x$ intersect at the point where $x = a$.

(a) Find the exact value of a, giving your answer in logarithmic form. [4]

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(b) Sketch C_1 and C_2 on the same diagram. [2]

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(c) Find the exact value of the length of the arc of C_1 from $x = 0$ to $x = a$. [5]

Q15.

6 The integral I_n, where n is an integer, is defined by $I_n = \int_0^{\frac{1}{2}} \left(1-x^2\right)^{-\frac{1}{2}n} \mathrm{d}x$.

(a) Find the exact value of I_1. [2]

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(b) By considering $\frac{\mathrm{d}}{\mathrm{d}x}\left(x\left(1-x^2\right)^{-\frac{1}{2}n}\right)$, or otherwise, show that

$$nI_{n+2} = 2^{n-1}3^{-\frac{1}{2}n} + (n-1)I_n.$$ [5]

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(c) Find the exact value of I_5 giving the answer in the form $k\sqrt{3}$, where k is a rational number to be determined. [3]

Q16.

2 Let $I_n = \int_0^1 (1+3x)^n e^{-3x} \, dx$, where n is an integer.

(a) Show that $3I_n = 1 - 4^n e^{-3} + 3nI_{n-1}$. [3]

(b) Find the exact value of I_2. [3]

Q17.

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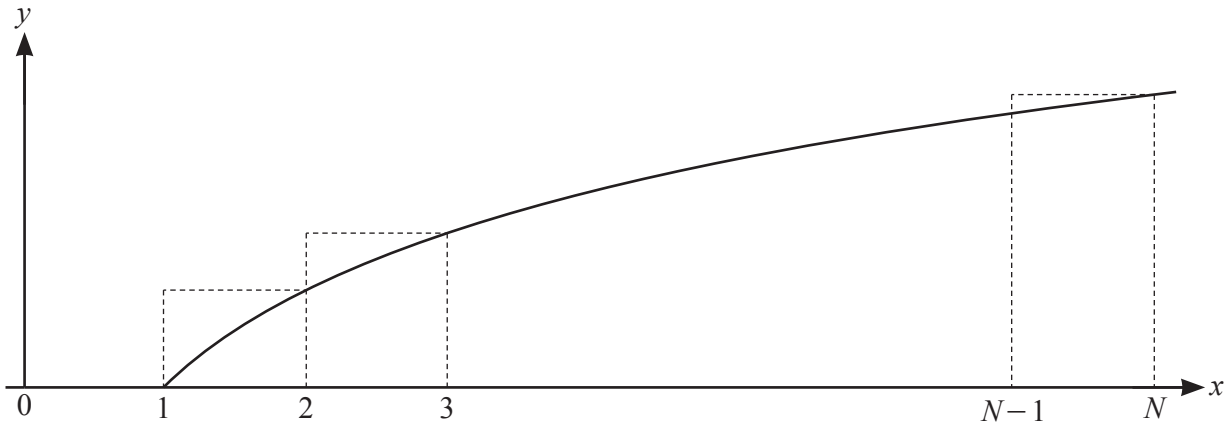

The diagram shows the curve with equation $y = \ln x$ for $x \geqslant 1$, together with a set of $(N-1)$ rectangles of unit width.

(a) By considering the sum of the areas of these rectangles, show that

$$\ln N! > N \ln N - N + 1. \qquad [5]$$

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(b) Use a similar method to find, in terms of N, an upper bound for $\ln N!$. [3]

Q18.

5 The curve C has parametric equations

$$x = \tfrac{1}{2}t^2 - \ln t, \qquad y = 2t + 1, \qquad \text{for } \tfrac{1}{2} \leqslant t \leqslant 2.$$

(a) Find the exact length of C. [5]

(b) Find $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ in terms of t, simplifying your answer. [4]

Q19

4 It is given that, for $n \geqslant 0$,

$$I_n = \int_0^1 x^n \mathrm{e}^{x^3} \, \mathrm{d}x.$$

(i) Show that $I_2 = \frac{1}{3}(\mathrm{e} - 1)$. [2]

(ii) Show that, for $n \geqslant 3$,

$$3I_n = \mathrm{e} - (n-2)I_{n-3}.$$ [3]

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(iii) Hence find the exact value of I_8. [3]

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Q20.

5 A curve C is defined parametrically by

$$x = \frac{2}{e^t + e^{-t}} \quad \text{and} \quad y = \frac{e^t - e^{-t}}{e^t + e^{-t}},$$

for $0 \leqslant t \leqslant 1$. The area of the surface generated when C is rotated through 2π radians about the x-axis is denoted by S.

(i) Show that $S = 4\pi \int_0^1 \frac{e^t - e^{-t}}{\left(e^t + e^{-t}\right)^2} \, dt$. [5]

(ii) Using the substitution $u = e^t + e^{-t}$, or otherwise, find S in terms of π and e. [3]

Q21.

10 Let $I_n = \int_{\frac{1}{4}\pi}^{\frac{1}{2}\pi} \cot^n x \, \mathrm{d}x$, where $n \geqslant 0$.

(i) By considering $\frac{\mathrm{d}}{\mathrm{d}x}(\cot^{n+1} x)$, or otherwise, show that

$$I_{n+2} = \frac{1}{n+1} - I_n. \qquad [5]$$

The curve C has equation $y = \cot x$, for $\frac{1}{4}\pi \leqslant x \leqslant \frac{1}{2}\pi$.

(ii) Find, in an exact form, the y-coordinate of the centroid of the region enclosed by C, the line $x = \frac{1}{4}\pi$ and the x-axis. [6]

Q22.

3 The integral I_n, where n is a positive integer, is defined by

$$I_n = \int_{\frac{1}{2}}^{1} x^{-n} \sin \pi x \, \mathrm{d}x.$$

(i) Show that

$$n(n+1)I_{n+2} = 2^{n+1}n + \pi - \pi^2 I_n. \quad [5]$$

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(ii) Find I_5 in terms of π and I_1. [2]

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Q23.

1 The curve C is defined parametrically by

$$x = e^t - t, \quad y = 4e^{\frac{1}{2}t}.$$

Find the length of the arc of C from the point where $t = 0$ to the point where $t = 3$. [5]

Q24.

9 **(i)** Using the substitution $u = \tan x$, or otherwise, find $\int \sec^2 x \tan^2 x \, \mathrm{d}x$. [2]

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It is given that, for $n \geqslant 0$,

$$I_n = \int_0^{\frac{1}{4}\pi} \sec^n x \tan^2 x \, \mathrm{d}x.$$

(ii) Using the result that $\frac{\mathrm{d}}{\mathrm{d}x}(\sec x) = \tan x \sec x$, show that, for $n \geqslant 2$,

$$(n+1)I_n = (\sqrt{2})^{n-2} + (n-2)I_{n-2}.$$ [5]

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(iii) Hence find the mean value of $\sec^4 x \tan^2 x$ with respect to x over the interval $0 \leqslant x \leqslant \frac{1}{4}\pi$, giving your answer in exact form. [3]

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Q25.

11 (i) Show that

$$\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} e^x \cos x \, dx = \frac{1}{2}\left(e^{\frac{1}{2}\pi} + e^{-\frac{1}{2}\pi}\right). \quad [4]$$

(ii) It is given that, for $n \geqslant 0$,

$$I_n = \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \mathrm{e}^{2x} \cos^n x \, \mathrm{d}x.$$

Show that, for $n \geqslant 2$,

$$4I_n = n(n-1) \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \mathrm{e}^{2x} \sin^2 x \cos^{n-2} x \, \mathrm{d}x - nI_n,$$

and deduce the reduction formula

$$(n^2 + 4)I_n = n(n-1)I_{n-2}. \qquad [6]$$

Q26.

4 A curve is defined parametrically by

$$x = t - \tfrac{1}{2}\sin 2t \quad \text{and} \quad y = \sin^2 t.$$

The arc of the curve joining the point where $t = 0$ to the point where $t = \pi$ is rotated through one complete revolution about the x-axis. The area of the surface generated is denoted by S.

(i) Show that

$$S = a\pi \int_0^{\pi} \sin^3 t \, \mathrm{d}t,$$

where the constant a is to be found. [5]

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(ii) Using the result $\sin 3t = 3\sin t - 4\sin^3 t$, find the exact value of S. [3]

Q27.

11 The curve C is defined parametrically by

$$x = 18t - t^2 \quad \text{and} \quad y = 8t^{\frac{3}{2}},$$

where $0 < t \leqslant 4$.

(i) Show that at all points of C,

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{3(9+t)}{2t^{\frac{1}{2}}(9-t)^3}. \qquad [4]$$

(ii) Show that the mean value of $\frac{d^2y}{dx^2}$ with respect to x over the interval $0 < x \leqslant 56$ is $\frac{3}{70}$. [4]

(iii) Find the area of the surface generated when C is rotated through 2π radians about the x-axis, showing full working. [6]

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Q28.

Let $I_n = \int_1^{\sqrt{2}} (x^2 - 1)^n \, dx$.

(i) Show that, for $n \geqslant 1$,

$$(2n + 1)I_n = \sqrt{2} - 2nI_{n-1}.$$ [5]

(ii) Using the substitution $x = \sec\theta$, show that

$$I_n = \int_0^{\frac{1}{4}\pi} \tan^{2n+1}\theta \sec\theta \, \mathrm{d}\theta. \qquad [4]$$

(iii) Deduce the exact value of

$$\int_0^{\frac{1}{4}\pi} \frac{\sin^7\theta}{\cos^8\theta}\,\mathrm{d}\theta.$$ [5]

Q29.

6 Let $I_n = \int_0^{\frac{1}{2}\pi} x^n \sin x \, \mathrm{d}x$.

(i) Prove that, for $n \geqslant 2$,

$$I_n + n(n-1)I_{n-2} = n\left(\tfrac{1}{2}\pi\right)^{n-1}. \qquad [4]$$

(ii) Calculate the exact value of I_1 and deduce the exact value of I_3. [3]

Q30.

12 The curve C has equation $y = \frac{1}{2}(e^x + e^{-x})$ for $0 \leqslant x \leqslant 4$.

(ii) Show that $\frac{ds}{dx} = \frac{1}{2}(e^x + e^{-x})$, where s denotes the arc length of C, and find the surface area generated when C is rotated through 2π radians about the x-axis. [4]

Q31.

5 A curve C has parametric equations

$$x = \tfrac{2}{5}t^{\frac{5}{2}} - 2t^{\frac{1}{2}}, \quad y = \tfrac{4}{3}t^{\frac{3}{2}}, \quad \text{for } 1 \leqslant t \leqslant 4.$$

(i) Find the exact value of the arc length of C. [5]

(ii) Find also the exact value of the surface area generated when C is rotated through 2π radians about the x-axis. [3]

Q32.

6 Let I_n denote $\int_0^2 (4 + x^2)^{-n}\,\mathrm{d}x$.

(i) Find $\frac{\mathrm{d}}{\mathrm{d}x}\left(x(4 + x^2)^{-n}\right)$ and hence show that

$$8nI_{n+1} = (2n - 1)I_n + 2 \times 8^{-n}. \qquad [5]$$

(ii) Use the result for integrating $\frac{1}{x^2 + a^2}$ with respect to x, in the List of Formulae (MF10), to find the value of I_1 and deduce that

$$I_3 = \tfrac{3}{1024}\pi + \tfrac{1}{128}.$$ [5]

Q33.

8 Let $I_n = \int_0^{\frac{1}{4}\pi} \sec^n x \, \mathrm{d}x$ for $n > 0$.

(i) Find the value of I_2. [2]

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(ii) Show that, for $n > 2$,

$$(n-1)I_n = 2^{\frac{1}{2}n-1} + (n-2)I_{n-2}.$$ [5]

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(iii) The curve C has equation $y = \sec^3 x$ for $0 \leqslant x \leqslant \frac{1}{4}\pi$. The region R is bounded by C, the x-axis, the y-axis and the line $x = \frac{1}{4}\pi$. Find the volume of revolution generated when R is rotated through 2π radians about the x-axis. [4]

Q34.

5 Let $I_n = \int_0^{\frac{1}{2}\pi} \cos^n x \sin^2 x \, \mathrm{d}x$, for $n \geqslant 0$. By differentiating $\cos^{n-1} x \sin^3 x$ with respect to x, prove that

$$(n+2)I_n = (n-1)I_{n-2} \quad \text{for } n \geqslant 2. \qquad [5]$$

Hence find the exact value of I_4. [4]

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Q35.

11 A curve C has parametric equations

$$x = \mathrm{e}^{2t}\cos 2t, \quad y = \mathrm{e}^{2t}\sin 2t, \quad \text{for } -\tfrac{1}{2}\pi \leqslant t \leqslant \tfrac{1}{2}\pi.$$

Find the arc length of C. [6]

Find the area of the surface generated when C is rotated through 2π radians about the x-axis. [8]

[Turn over

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Q36.

6 Let $I_n = \int_0^2 x^n(4 - x^2)^{\frac{1}{2}}\,\mathrm{d}x$, for $n \geqslant 1$. By considering $\frac{\mathrm{d}}{\mathrm{d}x}\left\{x^n(4 - x^2)^{\frac{3}{2}}\right\}$, show that

$$(n + 3)I_{n+1} = 4nI_{n-1}, \text{ where } n \geqslant 2. \quad [4]$$

Find the value of I_1 and deduce the exact value of I_3. [4]

Q37.

9 Evaluate $\int_0^{\frac{1}{2}\pi} x \sin x \, \mathrm{d}x$. [2]

Given that $I_n = \int_0^{\frac{1}{2}\pi} x^n \sin x \, \mathrm{d}x$, prove that, for $n > 1$,

$$I_n = n\left(\tfrac{1}{2}\pi\right)^{n-1} - n(n-1)I_{n-2}.$$ [4]

By first using the substitution $x = \cos^{-1} u$, find the value of

$$\int_0^1 \left(\cos^{-1} u\right)^3 \mathrm{d}u,$$

giving your answer in an exact form. [5]

Q38.

11 A curve C has parametric equations

$$x = 1 - 3t^2, \quad y = t(1 - 3t^2), \quad \text{for } 0 \leqslant t \leqslant \frac{1}{\sqrt{3}}.$$

Show that $\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = (1 + 9t^2)^2$. [2]

Hence find

(i) the arc length of C, [2]

(ii) the surface area generated when C is rotated through 2π radians about the x-axis. [3]

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Q39.

7 Let $I_n = \int_0^{\frac{1}{2}\pi} x^n \sin x \, \mathrm{d}x$, where n is a non-negative integer. Show that

$$I_n = n\left(\tfrac{1}{2}\pi\right)^{n-1} - n(n-1)I_{n-2}, \quad \text{for } n \geqslant 2. \qquad [5]$$

Find the exact value of I_4. [4]

Q40.

9 The curve C has parametric equations

$$x = 4t + 2t^{\frac{3}{2}}, \quad y = 4t - 2t^{\frac{3}{2}}, \quad \text{for } 0 \leqslant t \leqslant 4.$$

Find the arc length of C, giving your answer correct to 3 significant figures. [6]

Q41.

5 Let $I_n = \int_0^{\frac{1}{2}\pi} \frac{\sin 2n\theta}{\cos \theta} \, \mathrm{d}\theta$, where n is a non-negative integer.

(i) Use the identity $\sin P + \sin Q \equiv 2 \sin \frac{1}{2}(P + Q) \cos \frac{1}{2}(P - Q)$ to show that

$$I_n + I_{n-1} = \frac{2}{2n - 1}, \text{ for all positive integers } n. \quad [5]$$

(ii) Find the exact value of $\int_0^{\frac{1}{2}\pi} \frac{\sin 8\theta}{\cos \theta} \, \mathrm{d}\theta$. [4]

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Q42.

9 It is given that $I_n = \int_1^{\mathrm{e}} (\ln x)^n \, \mathrm{d}x$ for $n \geqslant 0$. Show that

$$I_n = (n-1)[I_{n-2} - I_{n-1}] \text{ for } n \geqslant 2. \qquad [6]$$

Hence find, in an exact form, the mean value of $(\ln x)^3$ with respect to x over the interval $1 \leqslant x \leqslant \mathrm{e}$. [6]

Q43.

8 The curve C has parametric equations

$$x = t^2, \quad y = t - \tfrac{1}{3}t^3, \quad \text{for } 0 \leqslant t \leqslant 1.$$

Find

(i) the arc length of C, [5]

(ii) the surface area generated when C is rotated through 2π radians about the x-axis. [5]

Q44.

10 It is given that $I_n = \int_0^{\frac{1}{4}\pi} \frac{\sin^{2n} x}{\cos x}\,\mathrm{d}x$, where $n \geqslant 0$. Show that

$$I_n - I_{n+1} = \frac{2^{-\left(n+\frac{1}{2}\right)}}{2n+1}. \qquad [5]$$

Hence show that $\int_0^{\frac{1}{4}\pi} \frac{\sin^6 x}{\cos x}\,\mathrm{d}x = \ln(1+\sqrt{2}) - \frac{73}{120}\sqrt{2}$. [5]

Q45.

6 The curve C has parametric equations

$$x = e^t - 4t + 3, \quad y = 8e^{\frac{1}{2}t}, \quad \text{for } 0 \leqslant t \leqslant 2.$$

(i) Find, in terms of e, the length of C. [5]

(ii) Find, in terms of π and e, the area of the surface generated when C is rotated through 2π radians about the x-axis. [5]

Q46.

9 Using the substitution $u = \cos\theta$, or any other method, find $\int \sin\theta\cos^2\theta\,\mathrm{d}\theta$. [1]

It is given that $I_n = \int_0^{\frac{1}{2}\pi} \sin^n\theta\cos^2\theta\,\mathrm{d}\theta$, for $n \geqslant 0$. Show that, for $n \geqslant 2$,

$$I_n = \frac{n-1}{n+2}I_{n-2}.$$ [5]

Hence find the exact value of $\int_0^{\frac{1}{2}\pi} \sin^4\theta\cos^2\theta\,\mathrm{d}\theta$. [4]

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[Turn over

Q47.

2 A curve C has parametric equations

$$x = e^t \cos t, \quad y = e^t \sin t, \quad \text{for } 0 \leqslant t \leqslant \tfrac{1}{2}\pi.$$

Find the arc length of C. [6]

Q48.

7 Let $I_n = \int_0^1 (1-x)^n e^x \, dx$. Show that, for all positive integers n,

$$I_n = nI_{n-1} - 1. \quad [3]$$

Find the exact value of I_4. [4]

By considering the area of the region enclosed by the x-axis, the y-axis and the curve with equation $y = (1-x)^4 e^x$ in the interval $0 \leqslant x \leqslant 1$, show that

$$\tfrac{65}{24} < e < \tfrac{11}{4}. \quad [3]$$

INTEGRATION
MS

Q1.

PUBLISHED

Question	Answer	Marks	Guidance
1	$s=\int_0^{\alpha}\sqrt{e^{\frac{1}{3}\theta}+\frac{9}{16}e^{\frac{1}{3}\theta}}\,d\theta=\frac{5}{4}\int_0^{\alpha}e^{\frac{1}{6}\theta}\,d\theta$	M1 A1	Forms s correctly.
	$s=\frac{5}{3}\left[e^{\frac{1}{6}\theta}\right]_0^{\alpha}=\frac{5}{3}\left(e^{\frac{1}{6}\alpha}-1\right)$	M1 A1	Integrates.
	$\alpha=\frac{6}{5}\ln\left(1+\frac{3}{5}s\right)$	A1	
		5	

Q2.

Question	Answer	Marks	Guidance
4(a)	$\int_0^1 2^x\,dx<\left(\frac{1}{N}\right)2^{\frac{1}{N}}+\left(\frac{1}{N}\right)2^{\frac{2}{N}}+\cdots+\left(\frac{1}{N}\right)2^{\frac{N-1}{N}}+\left(\frac{1}{N}\right)2^{\frac{N}{N}}$	M1 A1	Forms the sum of the areas of the rectangles.
	$=\frac{1}{N}\sum_{n=1}^{N}\left(2^{\frac{1}{N}}\right)^n=\frac{2^{\frac{1}{N}}}{N\left(2^{\frac{1}{N}}-1\right)}$	M1 A1	Applies $\sum_{n=1}^{N}r^n=\frac{r\left(r^N-1\right)}{r-1}$, AG.
		4	
4(b)	$\int_0^1 2^x\,dx>\left(\frac{1}{N}\right)+\left(\frac{1}{N}\right)2^{\frac{1}{N}}+\cdots+\left(\frac{1}{N}\right)2^{\frac{N-2}{N}}+\left(\frac{1}{N}\right)2^{\frac{N-1}{N}}$	M1 A1	Forms the sum of the areas of appropriate rectangles.
	$=\frac{1}{N}\sum_{n=0}^{N-1}\left(2^{\frac{1}{N}}\right)^n=\frac{1}{N\left(2^{\frac{1}{N}}-1\right)}$	M1 A1	Applies $\sum_{n=0}^{N-1}r^n=\frac{r^N-1}{r-1}$.
		4	
4(c)	$\frac{2^{\frac{1}{N}}}{N\left(2^{\frac{1}{N}}-1\right)}-\frac{1}{N\left(2^{\frac{1}{N}}-1\right)}=\frac{1}{N}<10^{-4}$ leading to $N>10^4$	M1	Simplifies U_n-L_n to $\frac{c}{n}$.
	Least value of N is 10001	A1	
		2	

Q3.

Question	Answer	Marks	Guidance
6(a)	$\int_0^1 \ln(1+x)\,dx < \left(\frac{1}{n}\right)\ln\left(1+\frac{1}{n}\right)+\left(\frac{1}{n}\right)\ln\left(1+\frac{2}{n}\right)+\cdots+\left(\frac{1}{n}\right)\ln\left(1+\frac{n}{n}\right)$	**M1 A1**	Forms the sum of the areas of the rectangles.
	$=\frac{1}{n}\left(n\ln\frac{1}{n}+\ln(n+1)+\ln(n+2)+\cdots+\ln(2n)\right)=\frac{1}{n}\ln\frac{(2n)!}{n!}-\ln n$	**M1 A1**	Applies laws of logarithms, AG.
		4	
6(b)	$\int_0^1 \ln(1+x)\,dx > \left(\frac{1}{n}\right)\ln\left(1+\frac{1}{n}\right)+\left(\frac{1}{n}\right)\ln\left(1+\frac{2}{n}\right)+\cdots+\left(\frac{1}{n}\right)\ln\left(1+\frac{n-1}{n}\right)$	**M1 A1**	Forms the sum of the areas of appropriate rectangles.
	$=\frac{1}{n}\left((n-1)\ln\frac{1}{n}+\ln(n+1)+\ln(n+2)+\cdots+\ln(2n-1)\right)$	**M1**	Applies laws of logarithms.
	$\frac{1}{n}\ln\frac{(2n-1)!}{n!}-\left(\frac{n-1}{n}\right)\ln n$	**A1**	Equivalent answers: $\frac{1}{n}\ln\frac{(2n)!}{n!}-\ln n-\frac{1}{n}\ln 2$; $\frac{1}{n}\ln\frac{(2n-1)!}{(n-1)!}-\ln n$.
		4	
6(c)	$U_n-L_n=\frac{\ln 2}{n}$	**M1**	Simplifies their U_n-L_n to $\frac{c}{n}$.
	$\frac{\ln 2}{n}\to 0$ as $n\to\infty$	**A1**	AG
		2	

Q4.

Question	Answer	Marks	Guidance
8(a)	$\int \sin\theta \cos^{n}\theta \, d\theta = -\frac{\cos^{n+1}\theta}{n+1} (+C)$	**M1 A1**	Allow with "$+C$" missing.
		2	
8(b)	$\int_0^{\frac{1}{2}\pi} \sin^{m-1}\theta \sin\theta \cos^{n}\theta \, d\theta = \left[-\frac{\sin^{m-1}\theta \cos^{n+1}\theta}{n+1} \right]_0^{\frac{1}{2}\pi} + \frac{m-1}{n+1} \int_0^{\frac{1}{2}\pi} \cos^{n+2}\theta \sin^{m-2}\theta \, d\theta$	**M1 A1**	Integrates by parts using part **(a)**. Allow with limits missing.
	$= \frac{m-1}{n+1} \int_0^{\frac{1}{2}\pi} \cos^{n}\theta \sin^{m-2}\theta (1-\sin^2\theta) \, d\theta$	**M1**	Applies $\cos^2\theta = 1-\sin^2\theta$.
	$I_{m,n} = \frac{m-1}{n+1} I_{m-2,n} - \frac{m-1}{n+1} I_{m,n}$	**M1**	Combines with LHS.
	$\frac{n+m}{n+1} I_{m,n} = \frac{m-1}{n+1} I_{m-2,n}$ leading to $I_{m,n} = \frac{m-1}{m+n} I_{m-2,n}$.	**A1**	AG.

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Question	Answer	Marks	Guidance
8(b)	**Alternative method for question 8(b)**		
	$\int_0^{\frac{1}{2}\pi} \sin^{m-2}\theta \sin^2\theta \cos^n\theta \,d\theta = \int_0^{\frac{1}{2}\pi} \sin^{m-2}\theta\left(1-\cos^2\theta\right)\cos^n\theta \,d\theta$	**M1**	Applies $\sin^2\theta = 1-\cos^2\theta$
	$= \int_0^{\frac{1}{2}\pi} \sin^{m-2}\theta \cos^n\theta \,d\theta - \int_0^{\frac{1}{2}\pi} \sin^{m-2}\theta \cos^{n+2}\theta \,d\theta$		
	$= I_{m-2,n} - \int_0^{\frac{1}{2}\pi} \sin^{m-2}\theta \cos\theta \cos^{n+1}\theta \,d\theta$		
	$= I_{m-2,n} - \left[\frac{1}{m-1}\sin^{m-1}\theta \cos^{n+1}\theta\right]_0^{\frac{1}{2}\pi} - \frac{n+1}{m-1}\int_0^{\frac{1}{2}\pi} \cos^n\theta \sin^m\theta \,d\theta$	**M1 A1**	$u = \cos^{n+1}\theta \Rightarrow u' = -(n+1)\cos^n\theta\sin\theta$ $v' = \sin^{m-2}\theta\cos\theta \Rightarrow v = \frac{1}{m-1}\sin^{m-1}\theta$
	$I_{m,n} = I_{m-2,n} - \frac{n+1}{m-1}I_{m,n} \Rightarrow \left(1+\frac{n+1}{m-1}\right)I_{m,n} = I_{m-2,n}$	**M1 A1**	Combines with LHS. AG
		5	
8(c)	$z + z^{-1} = 2\cos\theta$	**B1**	
	$\left(z+z^{-1}\right)^5 = \left(z^5+z^{-5}\right) + 5\left(z^3+z^{-3}\right) + 10\left(z+z^{-1}\right)$	**M1 A1**	Expands and groups. A0 if no grouping shown.
	$2^5\cos^5\theta = 2\cos 5\theta + 5\left(2\cos 3\theta\right) + 10\left(2\cos\theta\right)$	**M1**	Substitutes $z^n + z^{-n} = 2\cos n\theta$.
	$\cos^5\theta = \frac{1}{16}\cos 5\theta + \frac{5}{16}\cos 3\theta + \frac{5}{8}\cos\theta$	**A1**	
		5	

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Question	Answer	Marks	Guidance
8(d)	$I_{2,5} = \frac{1}{7} I_{0,5}$	**M1**	Applies reduction formula in part **(b).**
	$\frac{1}{7} I_{0,5} = \frac{1}{7}\int_0^{\frac{1}{2}\pi} \cos^5\theta \, d\theta = \frac{1}{7}\int_0^{\frac{1}{2}\pi} \frac{1}{16}\cos 5\theta + \frac{5}{16}\cos 3\theta + \frac{5}{8}\cos\theta \, d\theta$	**M1**	Applies part **(c)**.
	$= \frac{1}{7}\left[\frac{1}{80}\sin 5\theta + \frac{5}{48}\sin 3\theta + \frac{5}{8}\sin\theta\right]_0^{\frac{1}{2}\pi}$	**A1**	
	$= \frac{8}{105}$	**A1**	
		4	

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Q5.

Question	Answer	Marks	Guidance
3(a)	$\int_0^1 x^3\,dx < \left(\frac{1}{n}\right)\left(\frac{1}{n}\right)^3 + \left(\frac{1}{n}\right)\left(\frac{2}{n}\right)^3 + \cdots + \left(\frac{1}{n}\right)\left(\frac{n-1}{n}\right)^3 + \left(\frac{1}{n}\right)\left(\frac{n}{n}\right)^3$	**M1 A1**	Forms the sum of the areas of the n rectangles. Need last term for A1.
	$\frac{1}{n^4}\sum_{r=1}^{n} r^3 = \frac{n^2(n+1)^2}{4n^4} = \left(\frac{n+1}{2n}\right)^2$	**M1 A1**	Applies $\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$, AG.
		4	
3(b)	$\int_0^1 x^3\,dx > \left(\frac{1}{n}\right)\left(\frac{1}{n}\right)^3 + \left(\frac{1}{n}\right)\left(\frac{2}{n}\right)^3 + \cdots + \left(\frac{1}{n}\right)\left(\frac{n-1}{n}\right)^3$	**M1 A1**	Forms the sum of the areas of appropriate rectangles. Need the last term for A1.
	$\frac{1}{n^4}\sum_{r=1}^{n-1} r^3 = \frac{(n-1)^2 n^2}{4n^4} = \left(\frac{n-1}{2n}\right)^2$	**M1 A1**	Applies $\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$. Accept $\left(\frac{n+1}{2n}\right)^2 - \frac{1}{n}$.
		4	
3(c)	$\left(\frac{n+1}{2n}\right)^2 - \left(\frac{n-1}{2n}\right)^2 = \frac{1}{n} < 10^{-3}$ leading to $n > 10^3$	**M1**	Simplifies their $U_n - L_n$ to $\frac{k}{n}$.
	Least value of n is 1001.	**A1**	
		2	

Q6.

Question	Answer	Marks	Guidance
8(a)	$\frac{dx}{dt} = 2\sinh t$	**B1**	
	$\frac{dy}{dt} = \frac{3}{2} - \frac{1}{2}\cosh 2t = 1 - \left(\frac{1}{2}\cosh 2t - \frac{1}{2}\right) = 1 - \sinh^2 t$	**M1 A1**	Applies $2\sinh^2 t = \cosh 2t - 1$, AG.
		3	
8(b)(i)	$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 4s^2 + (s^2-1)^2 = 4s^2 + s^4 - 2s^2 + 1 = (s^2+1)^2$	**M1**	Factorises $\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2$.
	$\cosh^4 t$	**A1**	
	$2\pi\int_0^1 \left(\frac{3}{2}t - \frac{1}{4}\sinh 2t\right)\cosh^2 t \ dt = \pi\int_0^1 \left(\frac{3}{2}t - \frac{1}{4}\sinh 2t\right)(\cosh 2t + 1)\,dt$	**M1 A1**	Correct formula for surface area, AG. A0 if limits missing. $\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2$ does not need to be simplified for M1.
		4	

Question	Answer	Marks	Guidance
8(b)(ii)	$\int_0^1 \frac{1}{4}\sinh 2t(\cosh 2t + 1)\,dt = \left[\frac{1}{16}(1+\cosh 2t)^2\right]_0^1 = \frac{1}{16}(1+\cosh 2)^2 - \frac{1}{4}$	**M1 A1**	Integrates.
	$\frac{3}{2}\int_0^1 t(\cosh 2t + 1)\,dt = \frac{3}{2}\left[t\left(\frac{1}{2}\sinh 2t + t\right)\right]_0^1 - \frac{3}{2}\int_0^1 \frac{1}{2}\sinh 2t + t\,dt$	**M1 A1**	Integrates by parts.
	$\frac{3}{2}\left[t\left(\frac{1}{2}\sinh 2t + t\right) - \frac{1}{4}\cosh 2t - \frac{1}{2}t^2\right]_0^1 = \frac{3}{2}\left(\frac{1}{2}\sinh 2 - \frac{1}{4}\cosh 2 + \frac{3}{4}\right)$	**A1**	
	$\pi\left(\frac{3}{4}\sinh 2 - \frac{3}{8}\cosh 2 - \frac{1}{16}(1+\cosh 2)^2 + \frac{11}{8}\right)$	**A1**	OE. Must be exact. (Decimal answer is 3.980131435…)
		6	

Question	Answer	Marks	Guidance
3(a)	$\sum_{r=1}^{N}\frac{r}{2r^2-1}=1+\sum_{r=2}^{N}\frac{r}{2r^2-1}$	**M1 A1**	Compares with sum of the areas of the rectangles.
	$<\ 1+\int_1^N\frac{x}{2x^2-1}dx$	**M1**	Compares with integral.
	$\int_1^N\frac{x}{2x^2-1}dx=\left[\frac{1}{4}\ln\left(2x^2-1\right)\right]_1^N$	**M1 A1**	Finds integral.
	$\sum_{r=1}^{N}\frac{r}{2r^2-1}<1+\frac{1}{4}\ln\left(2N^2-1\right)$	**M1 A1**	Inserts limits, AG.
		7	
3(b)	$\sum_{r=1}^{N}\frac{r}{2r^2-1}=\sum_{r=1}^{N-1}\frac{r}{2r^2-1}+\frac{N}{2N^2-1}>\int_1^N\frac{x}{2x^2-1}dx+\frac{N}{2N^2-1}$	**M1 A1**	Compares with integral.
	$\frac{1}{4}\ln\left(2N^2-1\right)+\frac{N}{2N^2-1}$	**A1**	CWO.
	Alternative method for question 3(b)		
	$\sum_{r=1}^{N}\frac{r}{2r^2-1}>\int_1^{N+1}\frac{x}{2x^2-1}dx$	**M1 A1**	Compares with integral.
	$\frac{1}{4}\ln\left(2\left(N+1\right)^2-1\right)$	**A1**	CWO.
		3	

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Question	Answer	Marks	Guidance
7(a)	$I_1 = \int_0^{\frac{3}{2}} \left(4+x^2\right)^{-\frac{1}{2}} = \left[\sinh^{-1}\left(\frac{1}{2}x\right)\right]_0^{\frac{3}{2}}$	**M1 A1**	Recognises integral or uses appropriate substitution.
	$\ln 2$	**A1**	Insert limits, must simplify.
		3	
7(b)	$\frac{d}{dx}\left(x\left(4+x^2\right)^{-\frac{1}{2}n}\right) = -nx^2\left(4+x^2\right)^{-\frac{1}{2}n-1} + \left(4+x^2\right)^{-\frac{1}{2}n}$	**M1 A1**	Uses the product rule to differentiate.
	$-n\left(4+x^2-4\right)\left(4+x^2\right)^{-\frac{1}{2}n-1} + \left(4+x^2\right)^{-\frac{1}{2}n}$	**M1**	Applies $x^2 = 4+x^2-4$.
	$\left[x\left(4+x^2\right)^{-\frac{1}{2}n}\right]_0^{\frac{3}{2}} = -nI_n + 4nI_{n+2} + I_n$	**M1**	Integrates both sides using the limits given.
	$\frac{3}{2}\left(\frac{2}{5}\right)^n = (1-n)I_n + 4nI_{n+2} \Rightarrow 4nI_{n+2} = \frac{3}{2}\left(\frac{2}{5}\right)^n + (n-1)I_n$	**A1**	Substitutes limits and rearranges, AG.
	Alternative method for question 7(b)		
	$I_n = \left[x\left(4+x^2\right)^{-\frac{1}{2}n}\right]_0^{\frac{3}{2}} + n\int_0^{\frac{3}{2}} x^2\left(4+x^2\right)^{-\frac{1}{2}n-1}\,dx$	**M1 A1**	Integrates by parts.
	$I_n = \left[x\left(4+x^2\right)^{-\frac{1}{2}n}\right]_0^{\frac{3}{2}} + n\int_0^{\frac{3}{2}} \left(4+x^2-4\right)\left(4+x^2\right)^{-\frac{1}{2}n-1}\,dx$	**M1**	Applies $x^2 = 4+x^2-4$.
	$I_n = \frac{3}{2}\left(\frac{2}{5}\right)^n + nI_n - 4I_{n+2} \Rightarrow 4nI_{n+2} = \frac{3}{2}\left(\frac{2}{5}\right)^n + (n-1)I_n$	**M1 A1**	Substitutes limits and rearranges, AG.
		5	

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Question	Answer	Marks	Guidance
7(c)	$I_3 = \frac{3}{20}$	**B1**	Applies reduction formula with $n = 1$.
	$12I_5 = \frac{3}{2}\left(\frac{8}{125}\right) + 2I_3 \Rightarrow I_5 = \frac{33}{1000} = 0.033$	**M1 A1**	Applies reduction formula with $n = 3$.
		3	

Question	Answer	Marks	Guidance
4(a)	$\sum_{r=1}^{N}\frac{\ln r}{r^2}=\frac{\ln 2}{4}+\sum_{r=3}^{N}\frac{\ln r}{r^2}$	**M1 A1**	Compares with sum of the areas of the rectangles. M1 for writing out sum, A1 for considering $\sum_{r=3}^{N}\frac{\ln r}{r^2}$.
	$<\frac{\ln 2}{4}+\int_2^N\frac{\ln x}{x^2}dx$	**M1**	Compares with integral.
	$\int_2^N\frac{\ln x}{x^2}dx=\left[-\frac{\ln x+1}{x}\right]_2^N$	**M1 A1**	Finds integral.
	$\sum_{r=1}^{N}\frac{\ln r}{r^2}<\frac{\ln 2}{4}+\left(-\frac{\ln N+1}{N}+\frac{\ln 2+1}{2}\right)=\frac{2+3\ln 2}{4}-\frac{1+\ln N}{N}$	**M1 A1**	Inserts limits, AG. A1 requires second M1.
		7	
4(b)	$\sum_{r=1}^{N}\frac{\ln r}{r^2}=\sum_{r=2}^{N-1}\frac{\ln r}{r^2}+\frac{\ln N}{N^2}>\int_2^N\frac{\ln x}{x^2}dx+\frac{\ln N}{N^2}$	**M1 A1**	Compares with integral. Lower limit of 1 scores M0. Accept $\sum_{r=1}^{N}\frac{\ln r}{r^2}>\int_2^{N+1}\frac{\ln x}{x^2}$.
	$=\frac{\ln 2+1}{2}-\frac{\ln N+1}{N}+\frac{\ln N}{N^2}$	**A1**	Accept $\frac{\ln 2+1}{2}-\frac{\ln(N+1)+1}{N+1}$.
		3	

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Question	Answer	Marks	Guidance
8(a)	$\cosh A = \frac{1}{2}\left(e^{A} + e^{-A}\right)$	**B1**	Writes in exponential form
	$2\cosh^2 A = \frac{1}{2}\left(e^{2A} + 2 + e^{-2A}\right) = \frac{1}{2}\left(e^{2A} + e^{-2A}\right) + 1 = \cosh 2A + 1$	**M1 A1**	Expands, AG.
		3	
8(b)(i)	$\frac{dx}{dt} = 4\sinh 2t + 3 \quad \frac{dy}{dt} = 3\sinh 2t - 4$	**B1**	
	$(4\sinh 2t + 3)^2 + (3\sinh 2t - 4)^2 = 25(\sinh^2 2t + 1) = 25\cosh^2 2t$	**M1 A1**	Expands and applies $\cosh^2 A = \sinh^2 A + 1$
	$2\pi\int_{-\frac{1}{2}}^{\frac{1}{2}} (2\cosh 2t + 3t)(5\cosh 2t)\,dt \left[= \pi\int_{-\frac{1}{2}}^{\frac{1}{2}} 20\cosh^2 2t + 30t\cosh 2t\,dt\right]$	**A1**	Correct formula for surface area, AG.
		4	
8(b)(ii)	$20\int_{-\frac{1}{2}}^{\frac{1}{2}} \cosh^2 2t\,dt = 10\int_{-\frac{1}{2}}^{\frac{1}{2}} \cosh 4t + 1\,dt$	**M1**	Applies $\cosh^2 A = \frac{1}{2}(\cosh 2A + 1)$.
	$10\left[\frac{1}{4}\sinh 4t + t\right]_{-\frac{1}{2}}^{\frac{1}{2}} = 10\left(\frac{1}{2}\sinh 2 + 1\right)$	**M1 A1**	Integrates.
	$30\int_{-\frac{1}{2}}^{\frac{1}{2}} t\cosh 2t\,dt = 30\left(\frac{1}{2}\left[t\sinh 2t\right]_{-\frac{1}{2}}^{\frac{1}{2}} - \frac{1}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}} \sinh 2t\,dt\right)$	**M1 A1**	Integrates by parts.
	$15\left[t\sinh 2t - \frac{1}{2}\cosh 2t\right]_{-\frac{1}{2}}^{\frac{1}{2}} = 0$	**A1**	Accept $\int_{-\frac{1}{2}}^{\frac{1}{2}} t\cosh 2t\,dt = 0$ since $t\cosh 2t$ is odd.
	$10\pi\left(\frac{1}{4}\left(e^2 - e^{-2}\right) + 1\right)$	**A1**	64.82457...

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Question	Answer	Marks	Guidance
8(b)(ii)	**Alternative method for question 8(b)(ii)**		
	$10\pi\int_{-\frac{1}{2}}^{\frac{1}{2}}\left(2\cosh 2t+3t\right)\cosh 2t\,dt\ =$ $10\pi\left[\cosh 2t\sinh 2t+(\frac{3}{2}t\sinh 2t)\right]_{-\frac{1}{2}}^{\frac{1}{2}}-5\pi\int_{-\frac{1}{2}}^{\frac{1}{2}}\sinh 2t\left(4\sinh 2t+3\right)dt$	**M1 A1**	Integrates by parts. $u=2\cosh 2t+3t,\ u'=4\sinh 2t+3$ $v'=\cosh 2t,\ v=\frac{1}{2}\sinh 2t$
	$10\pi\left[\frac{1}{2}\sinh 4t\right]_{-\frac{1}{2}}^{\frac{1}{2}}-20\pi\int_{-\frac{1}{2}}^{\frac{1}{2}}\sinh^2 2t\,dt-\left(15\pi\int_{-\frac{1}{2}}^{\frac{1}{2}}\sinh 2t\,dt\right)=$ $10\pi\left[\frac{1}{2}\sinh 4t\right]_{-\frac{1}{2}}^{\frac{1}{2}}-10\pi\int_{-\frac{1}{2}}^{\frac{1}{2}}\cosh 4t-1\,dt-\left(15\pi\int_{-\frac{1}{2}}^{\frac{1}{2}}\sinh 2t\,dt\right)$	**M1 A1**	Applies $2\sinh^2 2t=\cosh 4t-1$
	$\pi\left[\frac{5}{2}\sinh 4t-\frac{10}{4}\sinh 4t+10t-\left(\frac{15}{2}\cosh 2t\right)\right]_{-\frac{1}{2}}^{\frac{1}{2}}$	**M1 A1**	Integrates
	$\pi\left[\frac{5}{2}\sinh 4t+10t-\left(\frac{15}{2}\cosh 2t\right)\right]_{-\frac{1}{2}}^{\frac{1}{2}}=\pi\left(5\sinh 2+10\right)=10\pi\left(\frac{1}{4}\left(e^2-e^{-2}\right)+1\right)$	**A1**	64.82457...
		7	

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Q11.

Question	Answer	Marks	Guidance
3(a)	$\int_0^1 1-x^2\,dx < \frac{1}{n}+\frac{1}{n}\left(1-\frac{1}{n^2}\right)+\ldots+\frac{1}{n}\left(1-\frac{(n-1)^2}{n^2}\right)$	**M1 A1**	Forms the sum of the areas of the rectangles. Three terms, including last for A1. Allow use of sigma.
	$1-\frac{1}{n^3}\sum_{r=1}^{n-1} r^2 = 1-\frac{(n-1)n(2n-1)}{6n^3}$	**M1**	Applies $\sum_{r=1}^{n-1} r^2 = \frac{1}{6}(n-1)n(2n-1)$.
	$\frac{6n^2-(n-1)(2n-1)}{6n^2} = \frac{4n^2+3n-1}{6n^2}$	**A1**	AG. [Can get this A1 without previous A1.]
		4	

Question	Answer	Marks	Guidance
3(a)	$\int_0^1 1-x^2\,dx < \frac{1}{n}+\frac{1}{n}\left(1-\frac{1}{n^2}\right)+\ldots+\frac{1}{n}\left(1-\frac{(n-1)^2}{n^2}\right)$	**M1 A1**	Forms the sum of the areas of the rectangles. Three terms, including last for A1. Allow use of sigma.
	$1-\frac{1}{n^3}\sum_{r=1}^{n-1} r^2 = 1-\frac{(n-1)n(2n-1)}{6n^3}$	**M1**	Applies $\sum_{r=1}^{n-1} r^2 = \frac{1}{6}(n-1)n(2n-1)$.
	$\frac{6n^2-(n-1)(2n-1)}{6n^2} = \frac{4n^2+3n-1}{6n^2}$	**A1**	AG. [Can get this A1 without previous A1.]
		4	

Question	Answer	Marks	Guidance
8(a)	$\tanh x = \dfrac{e^x - e^{-x}}{e^x + e^{-x}} \quad \operatorname{sech} x = \dfrac{2}{e^x + e^{-x}}$	**B1**	Writes in exponential form.
	$1 - \left(\dfrac{e^x - e^{-x}}{e^x + e^{-x}}\right)^2 = \dfrac{\left(e^x + e^{-x}\right)^2 - \left(e^x - e^{-x}\right)^2}{\left(e^x + e^{-x}\right)^2} = \dfrac{4}{\left(e^x + e^{-x}\right)^2}$	**M1**	Writes over common denominator.
	$\operatorname{sech}^2 x$	**A1**	AG
		3	
8(b)	$\int u^2 \, du = \frac{1}{3}\tanh^3 x + C$	**M1 A1**	Uses substitution correctly. Allow by inspection.
		2	
8(c)	$I_n = \int_0^{\ln 3} \operatorname{sech}^{n-2} x \operatorname{sech}^2 x \tanh^2 x \, dx$	**B1**	Separates into correct structure.
	$\left[\frac{1}{3}\operatorname{sech}^{n-2} x \tanh^3 x\right]_0^{\ln 3} + \frac{1}{3}(n-2)\int_0^{\ln 3} \operatorname{sech}^{n-2} x \tanh^4 x \, dx$	**M1**	Uses integration by parts correctly.
	$\frac{1}{3}\left(\frac{3}{5}\right)^{n-2}\left(\frac{4}{5}\right)^3 + \frac{1}{3}(n-2)\left(I_{n-2} - I_n\right)$	**M1 A1**	Uses $1 - \operatorname{sech}^2 x = \tanh^2 x$.
	$(3 + n - 2)I_n = \left(\frac{3}{5}\right)^{n-2}\left(\frac{4}{5}\right)^3 + (n-2)I_{n-2}$ leading to $(n+1)I_n = \left(\frac{3}{5}\right)^{n-2}\left(\frac{4}{5}\right)^3 + (n-2)I_{n-2}$	**A1**	AG
		5	

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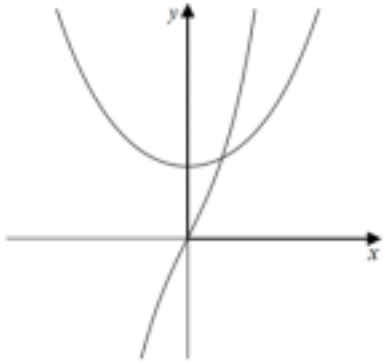
Question	Answer	Marks	Guidance
8(d)	$I_2 = \frac{1}{3}\left(\frac{4}{5}\right)^3 = \frac{64}{375}$	**B1**	
	$(4+1)I_4 = \left(\frac{3}{5}\right)^2\left(\frac{4}{5}\right)^3 + 2I_2$ leading to $I_4 = \frac{4928}{46875} = 0.105$	**M1 A1**	Applies reduction formula with $n = 4$.
		3	

Q 1 3 .

Question	Answer	Marks
4(a)	$\int_0^1 x^2\,dx < \left(\frac{1}{n}\right)\left(\frac{1}{n}\right)^2 + \left(\frac{1}{n}\right)\left(\frac{2}{n}\right)^2 + \cdots + \left(\frac{1}{n}\right)\left(\frac{n-1}{n}\right)^2 + \left(\frac{1}{n}\right)\left(\frac{n}{n}\right)^2$	**M1 A1**
	$\frac{1}{n^3}\sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6n^3} = \frac{2n^2+3n+1}{6n^2}$	**M1 A1**
		4
4(b)	$\int_0^1 x^2\,dx > \left(\frac{1}{n}\right)\left(\frac{1}{n}\right)^2 + \left(\frac{1}{n}\right)\left(\frac{2}{n}\right)^2 + \cdots + \left(\frac{1}{n}\right)\left(\frac{n-1}{n}\right)^2$	**M1 A1**
	$= \frac{1}{n^3}\sum_{r=1}^{n-1} r^2 = \frac{(n-1)(n)(2n-2+1)}{6n^3} = \frac{2n^2-3n+1}{6n^2}$	**M1 A1**
		4

Q14.

Question	Answer	Marks
5(a)	$\cosh a = 2\sinh a \cosh a \Rightarrow \sinh a = \frac{1}{2}$	**M1 A1**
	$a = \sinh^{-1}\frac{1}{2} = \ln\left(\frac{1}{2} + \sqrt{\frac{1}{4}+1}\right)$	**M1**
	$a = \ln\left(\frac{1}{2} + \frac{1}{2}\sqrt{5}\right)$	**A1**
		4

Question	Answer	Marks
5(b)		**B1**
	(**B1** for C_1 correct, **B1** for C_2 correct and intersecting C_1 in the first quadrant)	**B1**
		2
5(c)	$\int_0^a \sqrt{1+\sinh^2 x}\, dx$	**M1**
	$\int_0^a \sqrt{\cosh^2 x}\, dx = \int_0^a \cosh x\, dx$	**M1 A1**
	$\left[\sinh x\right]_0^a = \sinh a$	**M1**
	$\frac{1}{2}$	**A1**
		5

Question	Answer	Marks
6(a)	$I_1 = \int_0^{\frac{1}{2}} \left(1-x^2\right)^{-\frac{1}{2}} dx = \left[\sin^{-1} x\right]_0^{\frac{1}{2}} = \frac{1}{6}\pi$	**M1 A1**
		2
6(b)	$\frac{d}{dx}\left(x\left(1-x^2\right)^{-\frac{1}{2}n}\right) = nx^2\left(1-x^2\right)^{-\frac{1}{2}n-1} + \left(1-x^2\right)^{-\frac{1}{2}n}$	**M1 A1**
	$= n\left(1-\left(1-x^2\right)\right)\left(1-x^2\right)^{-\frac{1}{2}n-1} + \left(1-x^2\right)^{-\frac{1}{2}n}$	**M1**
	$\left[x\left(1-x^2\right)^{-\frac{1}{2}n}\right]_0^{\frac{1}{2}} = nI_{n+2} - nI_n + I_n$	**M1**
	$\frac{1}{2}\left(\frac{3}{4}\right)^{-\frac{1}{2}n} = nI_{n+2} - (n-1)I_n \Rightarrow nI_{n+2} = 2^{n-1}3^{-\frac{1}{2}n} + (n-1)I_n$	**A1**
		5
6(c)	$I_3 = 3^{-\frac{1}{2}}$	**B1**
	$3I_5 = 2^2 3^{-\frac{3}{2}} + 2I_3 \Rightarrow I_5 = \frac{10}{27}\sqrt{3}$	**M1 A1**
		3

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Q16.

Question	Answer	Marks
2(a)	$I_n = \left[-\frac{1}{3} e^{-3x} (1+3x)^n \right]_0^1 + n \int_0^1 e^{-3x} (1+3x)^{n-1} \, dx$	**M1 A1**
	$-\frac{1}{3} e^{-3} 4^n + \frac{1}{3} + nI_{n-1} \Rightarrow 3I_n = 1 - 4^n e^{-3} + 3nI_{n-1}$	**A1**
		3

Question	Answer	Marks
2(b)	$I_0 = \int_0^1 e^{-3x} \, dx = -\frac{1}{3} \left[e^{-3x} \right]_0^1 = \frac{1}{3} \left(1 - e^{-3} \right).$	**B1**
	$I_1 = \frac{1}{3} \left(1 - 4e^{-3} + 1 - e^{-3} \right) = \frac{1}{3} \left(2 - 5e^{-3} \right)$ $I_2 = \frac{1}{3} \left(1 - 16e^{-3} + 2 \left(2 - 5e^{-3} \right) \right)$	**M1**
	$I_2 = \frac{1}{3} \left(5 - 26e^{-3} \right)$	**A1**
		3

Q17.

Question	Answer	Marks
4(a)	$\ln 2 + \ln 3 + \ldots + \ln N$	**M1**
	$> \int_1^N \ln x \, dx$	**M1**
	$\int_1^N \ln x \, dx = \left[x \ln x \right]_1^N - \int_1^N 1 \, dx = N \ln N - N + 1$	**M1 A1**
	$\ln N! > N \ln N - N + 1$	**A1**
		5

Question	Answer	Marks
4(b)	$\ln 1 + \ln 2 + \ldots + \ln(N-1)$ (or $\ln 2 + \ldots + \ln(N-1)$)	**M1**
	$< \int_1^N \ln x \, dx$ (or $< \int_2^N \ln x \, dx$)	**M1**
	$\ln N! < N \ln N - N + 1 + \ln N = (N+1) \ln N - N + 1$ (or $\ln N! < (N+1) \ln N - N + 2(1 - \ln 2)$)	**A1**
		3

Q18.

Question	Answer	Marks
5(a)	$\dot{x}=t-t^{-1},\quad \dot{y}=2$	**B1**
	$\dot{x}^2+\dot{y}^2=t^2-2+t^{-2}+4=\left(t+t^{-1}\right)^2$	**M1 A1**
	$\int_{\frac{1}{2}}^{2} t+t^{-1}\,dt=\left[\frac{1}{2}t^2+\ln t\right]_{\frac{1}{2}}^{2}=\frac{15}{8}+2\ln 2$	**M1 A1**
		5

Question	Answer	Marks
5(b)	$\frac{dy}{dx}=\frac{\dot{y}}{\dot{x}}=\frac{2}{t-t^{-1}}$	**B1**
	$\frac{d}{dt}\left(\frac{2}{t-t^{-1}}\right)=\frac{-2\left(1+t^{-2}\right)}{\left(t-t^{-1}\right)^2}$	**B1**
	$\frac{d^2y}{dx^2}=\frac{d}{dt}\left(\frac{2}{t-t^{-1}}\right)\times\frac{dt}{dx}=\frac{-2\left(1+t^{-2}\right)}{\left(t-t^{-1}\right)^3}$	**M1 A1**
		4

Question	Answer	Marks	Guidance
4(i)	$\int_0^1 x^2 e^{x^3}\ dx = \frac{1}{3}\left[e^{x^3}\right]_0^1 = \frac{1}{3}(e-1)$	**M1 A1**	Must show working, AG.
		2	
4(ii)	$I_n = \int_0^1 x^{n-2} x^2 e^{x^3}\ dx = \left[\frac{1}{3}x^{n-2}e^{x^3}\right]_0^1 - \frac{n-2}{3}\int_0^1 x^{n-3}e^{x^3}\ dx$	**M1 A1**	Integrates by parts.
	$\frac{e}{3} - \frac{n-2}{3}I_{n-3} \Rightarrow 3I_n = e-(n-2)I_{n-3}$	**A1**	AG
		3	
4(iii)	$I_5 = \frac{1}{3}(e-3I_2) = \frac{1}{3}(e-e+1) = \frac{1}{3}$	**M1 A1**	Applies reduction formula once and uses **(i)**.
	$I_8 = \frac{1}{3}(e-6I_5) = \frac{1}{3}(e-2).$	**A1**	Must be exact.
		3	

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Question	Answer	Marks	Guidance
5(i)	$\dfrac{dx}{dt}=\dfrac{-2\left(e^{t}-e^{-t}\right)}{\left(e^{t}+e^{-t}\right)^{2}}$	**B1**	
	$\dfrac{dy}{dt}=\dfrac{\left(e^{t}+e^{-t}\right)^{2}-\left(e^{t}-e^{-t}\right)^{2}}{\left(e^{t}+e^{-t}\right)^{2}}=\dfrac{\left(2e^{t}\right)\left(2e^{-t}\right)}{\left(e^{t}+e^{-t}\right)^{2}}=\dfrac{4}{\left(e^{t}+e^{-t}\right)^{2}}$	**B1**	Differentiates and simplifies. Accept $1-\left(\dfrac{e^{t}-e^{-t}}{e^{t}+e^{-t}}\right)^{2}$.
	$\left(\dfrac{dx}{dt}\right)^{2}+\left(\dfrac{dy}{dt}\right)^{2}=\dfrac{4\left(e^{t}-e^{-t}\right)^{2}+16}{\left(e^{t}+e^{-t}\right)^{4}}=\dfrac{4\left(e^{t}+e^{-t}\right)^{2}}{\left(e^{t}+e^{-t}\right)^{4}}$	**M1 A1**	Attempt at writing $\left(\dfrac{dx}{dt}\right)^{2}+\left(\dfrac{dy}{dt}\right)^{2}$ as a square.
	$S=2\pi\displaystyle\int_{0}^{1}\left(\dfrac{e^{t}-e^{-t}}{e^{t}+e^{-t}}\right)\left(\dfrac{2}{e^{t}+e^{-t}}\right)dt=4\pi\int_{0}^{1}\dfrac{e^{t}-e^{-t}}{\left(e^{t}+e^{-t}\right)^{2}}dt$	**A1**	Uses correct formula, simplifies to AG.
		5	
5(ii)	$S=4\pi\displaystyle\int_{2}^{e+e^{-1}}u^{-2}\,du=4\pi\left[-u^{-1}\right]_{2}^{e+e^{-1}}$	**M1 A1**	Applies given substitution.
	$=4\pi\left(\dfrac{1}{2}-\dfrac{1}{e+e^{-1}}\right).$	**A1**	AEF, must be exact.
		3	

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Question	Answer	Marks	Guidance
10(i)	$\frac{d}{dx}\left(\cot^{n+1} x\right) = -(n+1)\cot^n x \csc^2 x$	**M1 A1**	Differentiates using chain rule.
	$= -(n+1)\cot^n x\left(\cot^2 x + 1\right)$ $= -(n+1)\cot^{n+2} x - (n+1)\cot^n x$	**M1**	Uses $\csc^2 x = \cot^2 x + 1$. OE
	$\left[\cot^{n+1} x\right]_{\frac{1}{4}\pi}^{\frac{1}{2}\pi} = -(n+1)\left(I_{n+2} + I_n\right)$	**M1**	Integrates both sides.
	$-1 = -(n+1)\left(I_{n+2} + I_n\right) \Rightarrow I_{n+2} = \frac{1}{n+1} - I_n$	**A1**	AG
10(i)	**Alternative method for 10(i)**		
	$\frac{d}{dx}\left(\frac{\cos^{n+1} x}{\sin^{n+1} x}\right) = \frac{-(n+1)\sin^{n+2} x\cos^n x - (n+1)\sin^n x\cos^{n+2} x}{\sin^{2n+2} x}$	**M1 A1**	Differentiates using quotient/chain rule.
	$-(n+1)\cot^n x - (n+1)\cot^{n+2} x$	**M1**	Separates
	$\left[\cot^{n+1} x\right]_{\frac{1}{4}\pi}^{\frac{1}{2}\pi} = -(n+1)\left(I_{n+2} + I_n\right)$	**M1**	Integrates both sides.
	$-1 = -(n+1)\left(I_{n+2} + I_n\right) \Rightarrow I_{n+2} = \frac{1}{n+1} - I_n$	**A1**	AG
		5	

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Question	Answer	Marks	Guidance
10(ii)	$\bar{y} = \frac{1}{2A}\int_{\frac{1}{4}\pi}^{\frac{1}{2}\pi} \cot^2 x\,dx = \frac{I_2}{2A}$	**M1**	Uses correct formula for $\bar{y}$. Allow incorrect or missing limits.
	$A = \int_{\frac{1}{4}\pi}^{\frac{1}{2}\pi} \cot x\,dx = \left[\ln \sin x\right]_{\frac{1}{4}\pi}^{\frac{1}{2}\pi} = \ln\sqrt{2}$	**M1 A1**	Integrates $\cot x$ to find area under the curve with correct limits.
	$I_2 = 1 - I_0 = 1 - \frac{1}{4}\pi$	**M1 A1**	Finds I_2
	$\bar{y} = \frac{1}{\ln 2}\left(1 - \frac{1}{4}\pi\right)$	**A1**	AEF. Must be exact.
		6	

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Question	Answer	Marks	Guidance
3(i)	$I_{n+2} = \left[\frac{x^{-n-1}}{-n-1}\sin \pi x\right]_{\frac{1}{2}}^{1} - \pi\int_{\frac{1}{2}}^{1}\frac{x^{-n-1}}{-n-1}\cos \pi x\,dx$	**M1 A1**	Integrates by parts.
	$= \frac{2^{n+1}}{n+1} + \frac{\pi}{n+1}\left(\left[\frac{x^{-n}}{-n}\cos \pi x\right]_{\frac{1}{2}}^{1} + \pi\int_{\frac{1}{2}}^{1}\frac{x^{-n}}{-n}\sin \pi x\,dx\right)$	**M1**	Integrates by parts again.
	$= \frac{2^{n+1}}{n+1} + \frac{\pi}{n+1}\left(\frac{1}{n} - \frac{\pi}{n}I_n\right)$ $\Rightarrow (n+1)I_{n+2} = 2^{n+1} + \pi\left(\frac{1}{n} - \frac{\pi}{n}I_n\right)$	**M1**	Uses I_n.
	$\Rightarrow n(n+1)I_{n+2} = 2^{n+1}n + \pi - \pi^2 I_n$	**A1**	AG
		5	
3(ii)	$2I_3 = 4 + \pi - \pi^2 I_1$ $12I_5 = 48 + \pi - \frac{\pi^2}{2}\left(4 + \pi - \pi^2 I_1\right)$	**M1**	Substitutes I_3 into reduction formula.
	$\Rightarrow I_5 = 4 + \frac{1}{24}\left(2\pi - 4\pi^2 - \pi^3 + \pi^4 I_1\right)$	**A1**	AEF, must be exact with fractions simplified.
		2	

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Q23.

Question	Answer	Marks	Guidance
1	$\frac{dx}{dt} = e^t - 1$ and $\frac{dy}{dt} = 2e^{\frac{1}{2}t}$	**B1**	
	$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \left(e^t + 1\right)^2$	**M1 A1**	M1 for using $\left(\frac{ds}{dt}\right)^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2$
	Arc length is $\int_0^3 e^t + 1 \, dt = \left[e^t + t\right]_0^3$	**M1**	M1 for good attempt at correct integral
	$= 2 + e^3$ (or 22.1)	**A1**	
		5	

Q24.

Question	Answer	Marks	Guidance
9(i)	$\frac{du}{dx} = \sec^2 x$ so integral becomes $\int u^2\,du$	**M1**	Uses substitution correctly
	$= \frac{1}{3}\tan^3 x + C$	**A1**	
		2	
9(ii)	$I_n = \int_0^{\frac{\pi}{4}} \sec^{n-2} x \sec^2 x \tan^2 x$	**B1**	Separates into correct structure
	$= \frac{1}{3}\left[\sec^{n-2} x \tan^3 x\right]_0^{\frac{\pi}{4}} - \frac{n-2}{3}\int_0^{\frac{\pi}{4}} \sec^{n-2} x \tan^4 x\,dx$	**M1**	Uses integration by parts correctly
	$= \frac{2^{\frac{n-2}{2}}}{3} - \frac{n-2}{3}\int_0^{\frac{\pi}{4}} \sec^{n-2} x \tan^2 x\left(\sec^2 x - 1\right)dx$	**M1**	Uses $\tan^2 x = \sec^2 x - 1$
	$= \frac{2^{\frac{n-2}{2}}}{3} - \frac{n-2}{3} I_n + \frac{n-2}{3} I_{n-2}$	**A1**	
	Thus $\left(3 + n - \right) I_n = \left(\sqrt{2}\right)^{n-2} + (n-2) I_{n-2}$.	**A1**	AG
		5	

Question	Answer	Marks	Guidance
9(iii)	$I_2 = \left[\frac{\tan^3 x}{3}\right]_0^{\frac{\pi}{4}} = \frac{1}{3}$	**B1 FT**	Attempts to find I_2
	$5I_4 = 2 + 2I_2$	**M1**	Uses reduction formula
	Mean value $= \frac{4}{\pi} I_4 = \frac{32}{15\pi}$	**A1**	
		3	

Q25.

Question	Answer	Marks	Guidance
11E(i)	$I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^x \cos x\,dx = \left[e^x \cos x\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^x(-)\sin x\,dx$	**M1 A1**	The parts can be either way round.
	$= \left[e^x \sin x\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} - I$	**M1**	Integrates by parts again.
	$\Rightarrow 2I = e^{\frac{\pi}{2}} + e^{-\frac{\pi}{2}}$ AG	**A1**	Alt method: Integrate $Re(e^{ix})e^x$ by parts M1, M1, A1, A1
11E(ii)	$I_n = \left[\frac{e^{2x}}{2}\cos^n x\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + \frac{n}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{2x}\cos^{n-1} x \sin x\,dx$	**M1**	Integrates by parts once.
	$= 0 - \frac{n}{4}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{2x}\left(\cos^n x - (n-1)\sin^2 x\cos^{n-2} x\right)dx$	**M1**	Integrates by parts again.
	$\Rightarrow 4I_n = n(n-1)\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{2x}\sin^2 x\cos^{n-2} x\,dx - nI_n$	**A1**	Simplifies to answer. (AG.)
	$(n+4)I_n = n(n-\ )\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{2x}\left(1-\cos^2 x\right)\cos^{n-2} x\,dx$	**M1**	For splitting $\sin^2 x$
	$\Rightarrow (n+4)I_n = n(n-1)I_{n-2} - n(n-1)I_n$	**M1**	
	$\Rightarrow (n^2+4)I_n = n(n-1)I_{n-2}$	**A1**	AG

Question	Answer	Marks	Guidance
4(i)	$\frac{dx}{dt} = 1 - \cos 2t$ $\frac{dy}{dt} = 2\sin t \cos t$	**B1**	B1 for $\frac{dx}{dt}$ and $\frac{dy}{dt}$.
	$\frac{dx}{dt} = 1 - \cos 2t = 2\sin^2 t$	**M1**	Uses an appropriate identity.
	$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{4\sin^4 t + 4\sin^2 t \cos^2 t} = 2\sin t$	**A1**	
	$S = 2\pi \int_0^{\pi} \left(\sin^2 t\right)\left(2\sin t\right) dt \Rightarrow a = 4$	**M1 A1**	Uses the correct formula for S
		5	
4(ii)	$S = 4\pi \int_0^{\pi} \sin^3 t \, dt = \pi \int_0^{\pi} 3\sin t - \sin 3t \, dt$	**M1**	Uses the result $4\sin^3 t = 3\sin t - \sin 3t$
	$= \pi \left[-3\cos t + \frac{1}{3}\cos 3t \right]_0^{\pi}$	**A1**	
	$= \pi \left(3 - \frac{1}{3} - \left(-3 + \frac{1}{3} \right) \right) = \frac{16\pi}{3}$	**A1**	Answer must be exact.
		3	

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Question	Answer	Marks	Guidance
11E(i)	$\frac{dy}{dx}=\frac{dy}{dt}\times\frac{dt}{dx}=\frac{12t^{\frac{1}{2}}}{18-2t}=\frac{6t^{\frac{1}{2}}}{9-t}$	**B1**	AEF.
	$\frac{d^2y}{dx^2}=\frac{d}{dt}\left(\frac{dy}{dx}\right)\times\frac{dt}{dx}$	**M1**	Uses chain rule again to find second derivative.
	$=\frac{3t^{-\frac{1}{2}}(9-t)+6t^{\frac{1}{2}}}{(9-t)^2(18-2t)}$	**dM1**	Uses quotient (or product rule)
	$=\frac{3t^{-\frac{1}{2}}(9-t+2t)}{2(9-t)^3}=\frac{3(9+t)}{2t^{\frac{1}{2}}(9-t)^3}$	**A1**	AG.
		4	
11E(ii)	$\frac{1}{56}\int_0^{56}\frac{d^2y}{dx^2}dx$	**B1**	Uses correct formula for mean value
	$=\frac{1}{56}\left[\frac{6t^{\frac{1}{2}}}{9-t}\right]_{t=0}^{t=4}$	**M1**	Finding limits correctly
		M1	Using expression
	$=\frac{1}{56}\left(\frac{6\sqrt{4}}{9-4}\right)=\frac{3}{70}.$	**A1**	Inserts correct values of t and obtains answer (AG.)
		4	

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Question	Answer	Marks	Guidance
11E(iii)	$\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2=(18-2t)^2+144t=4(t+9)^2$	**M1A1**	Simplifies $\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2$.
	$2\pi\int_0^4\left(8t^{\frac{3}{2}}\right)\left(2(t+9)\right)dt=32\pi\int_0^4 t^{\frac{5}{2}}+9t^{\frac{3}{2}}\,dt$	**M1A1**	Uses $2\pi\int y\frac{ds}{dt}dt$.
	$=32\pi\left[\frac{2}{7}t^{\frac{7}{2}}+\frac{18}{5}t^{\frac{5}{2}}\right]_0^4$	**M1**	Integrates term by term.
	$=\frac{2^{11}\times 83}{35}\pi=\frac{169984}{35}\pi=15300$	**A1**	Accept exact answer or decimal rounding to 15300.
		6	
Q28. 11O(i)	$I_n=\left[x\left(x^2-1\right)^n\right]_1^{\sqrt{2}}-2n\int_1^{\sqrt{2}}x^2\left(x^2-1\right)^{n-1}dx$	**M1A1**	Integrates by parts.
	$=\sqrt{2}-2n\int_1^{\sqrt{2}}\left(x^2-1+1\right)\left(x^2-1\right)^{n-1}dx$	**M1**	Uses $x^2=x^2-1+1$.
	$=\sqrt{2}-2nI_n-2nI_{n-1}$	**A1**	
	$\Rightarrow(2n+1)I_n=\sqrt{2}-2nI_{n-1}$	**A1**	AG.
		5	

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Question	Answer	Marks	Guidance
11O(ii)	$\frac{\mathrm{dx}}{\mathrm{d}\theta} = \tan\theta\sec\theta$	**M1A1**	Differentiates $\sec\theta$.
	$\sec\theta = \sqrt{2} \Rightarrow \theta = \frac{\pi}{4}$ $\sec\theta = 1 \Rightarrow \theta = 0$	**B1**	Changes limits.
	$\mathrm{I_n} = \int_0^{\frac{\pi}{4}} \left(\sec^2\theta - 1\right)^{\mathrm{n}} \tan\theta\sec\theta\,\mathrm{d}\theta = \int_0^{\frac{\pi}{4}} \tan^{2\mathrm{n}+1}\theta\sec\theta\,\mathrm{d}\theta$	**B1**	Uses $\sec^2\theta - 1 = \tan^2\theta$. (AG.)
		4	
11O(iii)	$\int_0^{\frac{\pi}{4}} \frac{\sin^7\theta}{\cos^8\theta}\,\mathrm{d}\theta = \mathrm{I_3}$	**B1**	Deduces that integral is $\mathrm{I_3}$.
	$\mathrm{I_0} = \sqrt{2} - 1$ (or $\mathrm{I_1} = \frac{2-\sqrt{2}}{3}$)	**B1**	Calculates $\mathrm{I_0}$ or $\mathrm{I_1}$
	$3\mathrm{I_1} = \sqrt{2} - 2\mathrm{I_0} \Rightarrow \mathrm{I_1} = \frac{2-\sqrt{2}}{3}.$	**M1A1**	Uses reduction formula to find $\mathrm{I_1}$ or $\mathrm{I_2}$ ……
	$\mathrm{I_2} = \frac{\sqrt{2}}{5} - \frac{4}{5}\mathrm{I_1} = \frac{7\sqrt{2}-8}{15}$		Finds $\mathrm{I_2}$.
	$\mathrm{I_3} = \frac{\sqrt{2}}{7} - \frac{6}{7}\mathrm{I_2} = \frac{16-9\sqrt{2}}{35}$	**A1**	Finds $\mathrm{I_3}$.
		5	

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Q29.

Question	Answer	Marks	Guidance
6(i)	$I_n = \left[-x^n \cos x\right]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} nx^{n-1} \cos x \, dx$	**M1A1**	Uses integration by parts with $u = x^n$
	$= 0 + \left[nx^{n-1} \sin x\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} n(n-1)x^{n-2} \sin x \, dx$	**A1**	
	$= n\left(\frac{1}{2}\pi\right)^{n-1} - n(n-1)I_{n-2} \Rightarrow I_n + n(n-1)I_{n-2} = n\left(\frac{1}{2}\pi\right)^{n-1}$ (AG)	**A1**	
	Total:	**4**	
6(ii)	$I_1 = \int_0^{\frac{\pi}{2}} x\sin x \, dx = \left[-x\cos x\right]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \cos x \, dx$	**M1**	
	$= \left[\sin x\right]_0^{\frac{\pi}{2}} = 1$	**A1**	
	$n = 3 \Rightarrow I_3 = 3\left(\frac{\pi}{2}\right)^2 - 3 \times 2 \times 1 = \frac{3}{4}\pi^2 - 6$	**B1 FT**	
	Total:	**3**	

Q30.

12E(ii)	$\frac{ds}{dx} = \sqrt{1 + (y')^2} = \sqrt{\frac{1}{4}\left(e^{2x} + 2 + e^{-2x}\right)} = \sqrt{\frac{1}{4}\left(e^x + e^{-x}\right)^2} = \frac{1}{2}\left(e^x + e^{-x}\right)$ (AG)	**B1**	
	$S = 2\pi\int_0^4 \frac{1}{4}\left(e^x + e^{-x}\right)^2 dx \quad = \frac{\pi}{2}\int_0^4 \left(e^{2x} + 2 + e^{-2x}\right)dx$	**M1 A1**	
	$= \frac{\pi}{2}\left[\frac{e^{2x}}{2} + 2x - \frac{e^{-2x}}{2}\right]_0^4 = \frac{\pi}{2}\left(\frac{e^8}{2} + 8 - \frac{e^{-8}}{2}\right)$	**A1**	
	Total:	**4**	

Q31.

Question	Answer	Marks	Guidance
5(i)	$\dot{x}^2 + \dot{y}^2 = \left(t^{\frac{3}{2}} - t^{-\frac{1}{2}}\right)^2 + 4t = \ldots = \left(t^{\frac{3}{2}} + t^{-\frac{1}{2}}\right)^2$	**M1A1**	SOI
	$s = \int_1^4 \left(t^{\frac{3}{2}} + t^{-\frac{1}{2}}\right) dt$	**M1**	
	$= \left[\frac{2}{5}t^{\frac{5}{2}} + 2t^{\frac{1}{2}}\right]_1^4$	**M1**	
	$= \left[\frac{64}{5} + 4\right] - \left[\frac{2}{5} + 2\right] \quad = \quad \frac{72}{5}$	**A1**	
	Total:	**5**	
5(ii)	$S = 2\pi \int_1^4 \frac{4}{3}t^{\frac{3}{2}}\left(t^{\frac{3}{2}} + t^{-\frac{1}{2}}\right) dt = \frac{8}{3}\pi \int_1^4 (t^3 + t) dt$	***M1**	
	$= \left(\frac{8}{3}\pi\right)\left[\frac{1}{4}t^4 + \frac{1}{2}t^2\right]_1^4$	**DM1**	
	$= \frac{8}{3}\pi\left\{\left[64 + 8\right] - \left[\frac{1}{4} + \frac{1}{2}\right]\right\} \quad = \quad 190\pi$	**A1**	
	Total:	**3**	

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Q32.

Question	Answer	Marks	Guidance
6(i)	$\frac{d}{dx}\left\{x\left(4+x^2\right)^{-n}\right\}=\left(4+x^2\right)^{-n}-2nx^2\left(4+x^2\right)^{-n-1}$	**M1A1**	
	Integrate w.r.t. x: $\left[x\left(4+x^2\right)^{-n}\right]_0^2=I_n-2n\int_0^2\left(4+x^2-4\right)\left(4+x^2\right)^{-n-1}dx$	**M1**	
		M1	
	$\Rightarrow 2.8^{-n}=I_n-2nI_n+8nI_{n+1}\Rightarrow 8nI_{n+1}=(2n-1)I_n+2.8^{-n}$	**A1**	AG
	Total:	**5**	
6(ii)	$I_1=\int_0^2\frac{1}{4+x^2}dx=\left[\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)\right]_0^2=\frac{1}{2}.\frac{\pi}{4}=\frac{\pi}{8}$	**M1A1**	
	$8I_2=\frac{\pi}{8}+\frac{1}{4}\Rightarrow I_2=\frac{\pi}{64}+\frac{1}{32}$	**M1A1FT**	
	$16I_3=\frac{3\pi}{64}+\frac{3}{32}+\frac{1}{32}\Rightarrow I_3=\frac{3\pi}{1024}+\frac{1}{128}$	**A1**	AG
	Total:	**5**	

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Question	Answer	Marks	Guidance
8(i)	$I_2 = \int_0^{\frac{1}{4}\pi} \sec^2 x dx = \left[\tan x\right]_0^{\frac{1}{4}\pi} = 1$	**M1A1**	
		2	
8(ii)	$I_n = \int_0^{\frac{1}{4}\pi} \sec^{n-2} x.\sec^2 x dx$	**M1**	
	$= \left[\sec^{n-2} x \tan x\right]_0^{\frac{1}{4}\pi} - \int_0^{\frac{1}{4}\pi} (n-2)\sec^{n-3} x(\sec x \tan x)\tan x dx$	**M1A1**	
	$= \left[\sec^{n-2} x \tan x\right]_0^{\frac{1}{4}\pi} - (n-2)\int_0^{\frac{1}{4}\pi} \sec^{n-2} x\left(\sec^2 x - 1\right)dx$	**M1A1**	
	$\Rightarrow (n-1)I_n = 2^{\frac{1}{2}n-1} + (n-2)I_{n-2}$		AG
		5	

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5	$\frac{\mathrm{d}}{\mathrm{d}x}\left(\cos^{n-1}x\sin^3 x\right) = -(n-1)\cos^{n-2}x\sin^4 x + 3\cos^n x\sin^2 x$	**M1A1**
	$\Rightarrow \left[\cos^{n-1}x\sin^3 x\right]_0^{\frac{1}{2}\pi} = -\int_0^{\frac{1}{2}\pi}(n-1)\cos^{n-2}x\sin^2 x\left(1-\cos^2 x\right)\mathrm{d}x + 3I_n$	**M1**
	$\Rightarrow 0 = -(n-1)I_{n-2} + (n-1)I_n + 3I_n$	**M1**
	$\Rightarrow (n+2)I_n = (n-1)I_{n-2}$ (**AG**)	**A1** [5]
	$I_0 = \int_0^{\frac{1}{2}\pi}\sin^2 x\mathrm{d}x = \frac{1}{2}\int_0^{\frac{1}{2}\pi}(1-\cos 2x)\mathrm{d}x$	**M1**
	$= \left[\frac{x}{2} - \frac{\sin 2x}{4}\right]_0^{\frac{1}{2}\pi} = \frac{\pi}{4}$	**A1**
	$I_2 = \frac{1}{4}\times\frac{\pi}{4}I_4 = \frac{1}{2}\times\frac{1}{4}\times\frac{\pi}{4} = \frac{\pi}{32}$	**M1A1** [4]

Qu	Solution	Part Marks
11 (e)	$\dot{x} = 2e^{2t}\cos 2t - 2e^{2t}\sin 2t \dot{y} = 2e^{2t}\cos 2t + 2e^{2t}\sin 2t$	B1
	$\dot{x}^2 + \dot{y}^2 = 4e^{4t}\left(\cos^2 2t - 2\cos 2t \sin 2t + \sin^2 2t + \cos^2 2t + 2\cos 2t \sin 2t + \sin^2 2t\right)$	M1
	$= 8e^{4t}$	A1
	$s = \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} 2\sqrt{2}e^{2t}\,dt$	M1
	$= \sqrt{2}\left[e^{2t}\right]_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} = \sqrt{2}\left(e^{\pi} - e^{-\pi}\right)$ or $2\sqrt{2}\sinh\pi$ or 32.7	M1A1 [6]
	$S = 2\pi\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} e^{2t}\sin 2t.2\sqrt{2}e^{2t}\,dt = 4\sqrt{2}\pi\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} e^{4t}\sin 2t\,dt$	M1
	Let $I = \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} e^{4t}\sin 2t\,dt = \left[-e^{4t}\frac{\cos 2t}{2}\right]_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} + \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} 2e^{4t}\cos 2t\,dt$	M1A1
	$= \left[\frac{e^{2\pi}}{2}\right] - \left[\frac{e^{-2\pi}}{2}\right] + 2\left\{\left[e^{4t}\frac{\sin 2t}{2}\right]_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} - \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} 2e^{4t}\sin 2t\right\}$	A1A1
	$= \frac{e^{2\pi} - e^{-2\pi}}{2} + 0 - 4I$	M1
	$\Rightarrow I = \frac{e^{2\pi} - e^{-2\pi}}{10}$	A1
	$\Rightarrow S = 4\sqrt{2}\pi.\frac{e^{2\pi} - e^{-2\pi}}{10} = \frac{2\sqrt{2}\pi}{5}\left(e^{2\pi} - e^{-2\pi}\right)$ or $\frac{4\sqrt{2}\pi}{5}\sinh 2\pi$ or 952 (3sf)	A1 [8]

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Q36.

Qu	Soluti	Marks
6	$\frac{\mathrm{d}}{\mathrm{d}x}\left\{x^n\left(4-x^2\right)^{\frac{3}{2}}\right\}=-3x^{n+1}\left(4-x^2\right)^{\frac{1}{2}}+nx^{n-1}\left(4-x^2\right)^{\frac{3}{2}}$	**B1**
	$-3x^{n+1}\left(4-x^2\right)^{\frac{1}{2}}+nx^{n-1}\left(4-x^2\right)^{\frac{1}{2}}\left(4-x^2\right)$	**M1**
	$\Rightarrow\left[x^n\left(4-x^2\right)^{\frac{3}{2}}\right]_0^2=0=-3I_{n+1}+4nI_{n-1}-nI_{n+1}$	**M1**
	$\Rightarrow(n+3)I_{n+1}=4nI_{n-1}$ **(AG)**	**A1** [4]
	$I_1=\left[-\frac{1}{3}\left(4-x^2\right)^{\frac{3}{2}}\right]_0^2=\frac{8}{3}$	**M1A1**
	$5I_3=8I_1\Rightarrow I_3=\frac{64}{15}$	**M1A1√** [4]

Q37.

9	$\int_0^{\frac{\pi}{2}} x\sin x\,dx = \left[-x\cos x\right]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \cos x\,dx = 0 + \left[\sin x\right]_0^{\frac{\pi}{2}} = 1$	**M1A1**	[2]
	$I_n = \left[-x^n \cos x\right]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} nx^{n-1}\cos x\,dx$	**M1A1**	
	$= \left[-x^n \cos x + nx^{n-1}\sin x\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} n(n-1)x^{n-2}\sin x\,dx$	**A1**	
	$= n\left(\frac{\pi}{2}\right)^{n-1} - n(n-1)I_{n-2}$	**A1**	[4]
	$x = \cos^{-1} u \Rightarrow u = \cos x \Rightarrow \frac{du}{dx} = -\sin x\ ;\ u = 0, 1 \Rightarrow x = \pi/2, 0$	**B1B1**	
	$\int_0^1 \left(\left[\cos^{-1} u\right]^3\right) du = -\int_{\pi/2}^0 x^3 \sin x\,dx = \int_0^{\pi/2} x^3 \sin x\,dx = I_3$	**B1**	
	$I_3 = 3\left(\frac{\pi}{2}\right)^2 - 3\times 2\times I_1 = \frac{3}{4}\pi^2 - 6$ (OE)	**M1A1**	[5]

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Q38.

11 O	$\dot{x}^2 + \dot{y}^2 = 36t^2 + 1 - 18t^2 + 81t^4 = 1 + 18t^2 + 81t^4 = \left(1 + 9t^2\right)^2 \qquad (\text{AG})$	**M1A1** [2]
(i)	$s = \int_0^{\frac{1}{\sqrt{3}}} \left(1 + 9t^2\right) dt = \left[t + 3t^3\right]_0^{\frac{1}{\sqrt{3}}} = \frac{2}{\sqrt{3}}$	**M1A1** [2]
(ii)	$S = 2\pi \int_0^{\frac{1}{\sqrt{3}}} t\left(1 - 3t^2\right)\left(1 + 9t^2\right) dt = 2\pi \int_0^{\frac{1}{\sqrt{3}}} \left(t + 6t^3 - 27t^5\right) dt$	***M1**
	$= 2\pi\left[\frac{1}{2}t^2 + \frac{3}{2}t^4 - \frac{9}{2}t^6\right]_0^{\frac{1}{\sqrt{3}}} = \frac{\pi}{3}$	**DM1A1** [3]

Q39.

7	$I_n = \int_0^{\frac{\pi}{2}} x^n \sin x \, dx = \left[-x^n \cos x\right]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} n x^{n-1} \cos x \, dx$ (LNR) $= 0 + \left[nx^{n-1} \sin x\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} n(n-1)x^{n-2} \sin x \, dx$ (LR) $= n\left(\frac{\pi}{2}\right)^{n-1} - n(n-1)I_{n-2}$ (AG) $I_0 = \int_0^{\frac{\pi}{2}} x^n \sin x \, dx = \left[-\cos x\right]_0^{\frac{\pi}{2}} = 1$ $I_2 = \pi - 2$, using reduction formula. $I_4 = 4 \times \left(\frac{\pi}{2}\right)^3 - 12(\pi - 2) = \frac{1}{2}\pi^3 - 12\pi + 24$ (If I_2 found by integration M1A1 (if correct) then M1A1 for I_4 from reduction formula.)	M1A1 M1A1 A1 (5) B1 M1A1 A1 (4) **Total: 9**

Q40.

9	$\dot{x} = 4 + 3t^{\frac{1}{2}} \quad \dot{y} = 4 - 3t^{\frac{1}{2}}$	B1
	$\dot{x}^2 + \dot{y}^2 = 32 + 18t$	B1
	$s = \int_0^4 (32+18t)^{\frac{1}{2}} \mathrm{d}t = \left[\frac{1}{27}(32+18t)^{\frac{3}{2}} \right]_0^4$	M1A1
	$= \frac{1}{27}\left(104^{\frac{3}{2}} - 32^{\frac{3}{2}} \right) = 32.6$ (CAO)	M1A1 (6)

Q41.

5	$I_n + I_{n-1} = \int_0^{\frac{\pi}{2}} \frac{\sin(2n\theta) + \sin(2n-2)\theta}{\cos\theta} \mathrm{d}\theta = \int_0^{\frac{\pi}{2}} \frac{2\sin(2n-1)\theta\cos\theta}{\cos\theta} \mathrm{d}\theta$ (LR)	M1A1
	$= \int_0^{\frac{\pi}{2}} 2\sin(2n-1)\theta \, \mathrm{d}\theta = \left[-\frac{2\cos(2n-1)\theta}{(2n-1)} \right]_0^{\frac{\pi}{2}}$ (LR)	dM1A1
	$= [0] - \left[-\frac{2}{(2n-1)} \right] = \frac{2}{2n-1}$ (AG)	A1 (5)
	$I_1 = \int_0^{\frac{\pi}{2}} \frac{\sin 2\theta}{\cos\theta} \mathrm{d}\theta = \int_0^{\frac{\pi}{2}} 2\sin\theta \, \mathrm{d}\theta = \left[-2\cos\theta \right]_0^{\frac{\pi}{2}} = [0] - [-2] = 2$	B1
	$I_2 = \frac{2}{3} - 2 = -\frac{4}{3}$	M1A1
	$\Rightarrow I_3 = \frac{2}{5} + \frac{4}{3} = \frac{26}{15} \Rightarrow I_4 = \frac{2}{7} - \frac{26}{15} = -\frac{152}{105}$	A1 (4) **Total 9**

Q42.

Qn & Part	Solution	Marks
9	$\int_1^e \ln x\,dx = x\ln x - x$	**B1**
	$I_n = \int_1^e (\ln x)^{n-1}.\ln x\,dx$	**M1**
	$= \left[(\ln x)^{n-1}(x\ln x - x)\right]_1^e - \int_1^e (n-1)(\ln x)^{n-2}.\frac{1}{x}(x\ln x - x)dx$	**M1A1**
	$= 0 - \int_1^e (n-1)(\ln x)^{n-2}(\ln x - 1)dx = (n-1)\left[I_{n-2} - I_{n-1}\right]$ (AG)	**M1A1** [6]
	Alternative for obtaining reduction formula:	
	$I_n = \int_1^e (\ln x)^n \times 1dx = \left[x(\ln x)^n\right]_1^e - \int_1^e n(\ln x)^{n-1}dx$	**M1A1**
	$\Rightarrow I_n = e - nI_{n-1}$	**A1**
	Similarly $I_{n-1} = e - (n-1)I_{n-2}$	**B1**
	$\Rightarrow I_n + nI_{n-1} = I_{n-1} + (n-1)I_{n-2}$	**M1**
	$\Rightarrow I_n = (n-1)\left[I_{n-2} - I_{n-1}\right]$ (AG)	**A1** [6]
	$I_0 = \left[x\right]_1^e = e - 1$	**B1**
	$I_1 = \left[x\ln x - x\right]_1^e = 1$	**B1**
	$I_2 = 1\times(e - 1 - 1) = e - 2$	**M1**
	$I_3 = 2(I_1 - I_2) = 2(1 - [e-2]) = 6 - 2e$	**A1**
	$\text{MV} = \frac{I_3}{e-1} = \frac{6-2e}{e-1}$	**M1 A1** [6] **Total 12**

Q43.

8	**(i**	$\dot{x} = 2t \quad \dot{y} = 1 - t^2$	B1
		$\dot{x}^2 + \dot{y}^2 = 4t^2 + 1 - 2t^2 + t^4 = (1+t^2)^2$ ACF	M1A1
		$s = \int_0^1 (1+t^2)\,dt = \left[t + \frac{t^3}{3}\right]_0^1 = \frac{4}{3}$ or 1.33 LR	M1A1 [5]
	(ii)	$S = 2\pi\int_0^1 \left(t - \frac{1}{3}t^3\right)\left(1+t^2\right)dt = 2\pi\int_0^1 \left(t + \frac{2}{3}t^3 - \frac{1}{3}t^5\right)dt$ LNR	M1A1
		$= 2\pi\left[\frac{1}{2}t^2 + \frac{1}{6}t^4 - \frac{1}{18}t^6\right]_0^1$ LR	M1A1
		$= \frac{11}{9}\pi$ or 3.84	A1 [5]

Q44.

10	$I_{n+1} = \int_0^{\frac{1}{4}\pi} \frac{\sin^{2n+2} x}{\cos x} dx$ (LR)	B1
	$I_{n+1} = \int_0^{\frac{1}{4}\pi} \frac{(1-\cos^2 x)\sin^{2n} x}{\cos x} dx$ (LR)	M1
	$= I_n - \int_0^{\frac{1}{4}\pi} \cos x \sin^{2n} x \, dx$	A1
	$= I_n - \left[\frac{\sin^{2n+1} x}{2n+1}\right]_0^{\frac{1}{4}\pi}$	M1
	$\Rightarrow I_n - 1_{n+1} = \frac{2^{-\left(n+\frac{1}{2}\right)}}{2n+1}$. (AG)	A1 [5]
	$I_0 = \int_0^{\frac{1}{4}\pi} \sec x \, dx = \left[\ln\left\|\sec x + \tan x\right\|\right]_0^{\frac{1}{4}\pi} = \ln\left(1+\sqrt{2}\right)$	M1A1
	$I_1 = \ln\left(1+\sqrt{2}\right) - \frac{1}{\sqrt{2}}$	M1
	$I_2 = \ln\left(1+\sqrt{2}\right) - \frac{1}{\sqrt{2}} - \frac{1}{6\sqrt{2}}$	A1
	$I_3 = \ln\left(1+\sqrt{2}\right) - \frac{1}{\sqrt{2}} - \frac{1}{6\sqrt{2}} - \frac{1}{20\sqrt{2}} = \ln\left(1+\sqrt{2}\right) - \frac{73\sqrt{2}}{120}$ (AG)	A1 [5]

Q45.

6	**(i)**	$\dot{x} = e^t - 4$, $\dot{y} = 4e^{\frac{1}{2}t}$ (both)	B1
		$\dot{x}^2 + \dot{y}^2 = e^{2t} - 8e^t + 16 + 16e^t = (e^t + 4)^2$ (ACF)	M1A1
		$s = \int_0^2 \left(e^t + 4\right) dt = \left[e^t + 4t\right]_0^2 = e^2 + 8 - 1 = e^2 + 7$	M1A1 [5]
	(ii)	$S = 2\pi \int_0^2 8e^{\frac{1}{2}t}\left(e^t + 4\right) dt$	B1√
		$16\pi \int_0^2 \left(e^{\frac{3}{2}t} + 4e^{\frac{1}{2}t}\right) dt = 16\pi \left[\frac{2}{3}e^{\frac{3}{2}t} + 8e^{\frac{1}{2}t}\right]_0^2$	M1A1
		$= 16\pi\left\{\left[\frac{2}{3}e^3 + 8e\right] - \left[\frac{2}{3} + 8\right]\right\} = 16\pi\left(\frac{2}{3}e^3 + 8e - \frac{26}{3}\right)$ (ACF)	M1A1 [5]
		(N.B. B1M1A1 can be earned without use of limits.)	

Q46.

9	$\int \sin\theta\cos^2\theta\,\mathrm{d}\theta = -\frac{1}{3}\cos^3\theta + c$ (Ignore omission of c.)	B1 [1]
	$I_n = \int_0^{\frac{\pi}{2}} \sin^n\theta\cos^2\theta\,\mathrm{d}\theta = \left[-\sin^{n-1}\theta.\frac{\cos^3\theta}{3}\right]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}}(n-1)\sin^{n-2}\theta\cos\theta.\frac{\cos^3\theta}{3}\mathrm{d}\theta$	M1A1
	$= 0 + \int_0^{\frac{\pi}{2}}\frac{(n-1)}{3}\sin^{n-2}\theta\cos^2\theta(1-\sin^2\theta)\,\mathrm{d}\theta$	M1A1
	(N.B. Limits not required for both M marks; also the parts for integration can be: $u = \cos x$ and $\frac{\mathrm{d}v}{\mathrm{d}x} = \sin^n x\cos x$.)	
	$= \frac{(n-1)}{3}(I_{n-2} - I_n)$	
	$\Rightarrow \ldots \Rightarrow I_n = \frac{(n-1)}{(n+2)}I_{n-2}$ (AG)	A1 [5]
	$I_0 = \int_0^{\frac{\pi}{2}}\cos^2\theta\,\mathrm{d}\theta = \frac{1}{2}\int_0^{\frac{\pi}{2}}(\cos 2\theta + 1)\mathrm{d}\theta = \frac{1}{2}\left[\frac{\sin 2\theta}{2} + \theta\right]_0^{\frac{\pi}{2}} = \frac{\pi}{4}$ (Or I_2)	M1A1
	$\therefore I_4 = \frac{3}{6}\times\frac{1}{4}\times\frac{\pi}{4} = \frac{\pi}{32}$ (CAO)	M1A1 [4]

Q47.

2	$\dot{x} = \mathrm{e}^t\cos t - \mathrm{e}^t\sin t \qquad \dot{y} = \mathrm{e}^t\sin t + \mathrm{e}^t\cos t$	M1A1
	$\dot{x}^2 + \dot{y}^2 = 2\mathrm{e}^{2t}\left(\cos^2 t + \sin^2 t\right) = 2\mathrm{e}^{2t}$	B1
	$s = \int_0^{\frac{1}{2}\pi}\sqrt{2}\mathrm{e}^t\mathrm{d}t$	M1
	$= \left[\sqrt{2}\mathrm{e}^t\right]_0^{\frac{1}{2}\pi} = \sqrt{2}\left(e^{\frac{1}{2}\pi} - 1\right) \quad (= 5.39)$	A1A1 (6) **[6]**

Q48.

7	$I_n = \left[(1-x)^n \mathrm{e}^x\right]_0^1 + \int_0^1 n(1-x)^{n-1}\mathrm{e}^x \mathrm{d}x$	M1A1
	$= 0 - 1 + nI_{n-1} = nI_{n-1} - 1$	A1 (3)
	$I_0 = \int_0^1 \mathrm{e}^x \mathrm{d}x = \left[e^x\right]_0^1 = \mathrm{e} - 1$	B1
	$I_4 = 4I_3 - 1 = 12I_2 - 5 = 24I_1 - 17 = 24I_0 - 41$	M1A1
	$= 24\mathrm{e} - 65$	A1
	Or for the last 3 marks: $I_1 = \mathrm{e} - 2$, $I_2 = 2\mathrm{e} - 5$	(M1A1)
	$I_3 = 6\mathrm{e} - 16$ and $I_4 = 24\mathrm{e} - 65$	(A1) (4)
	Draws appropriate sketch, or explains clearly. (Must be seen.)	B1
	$0 <$ Area below graph $< 1 \Rightarrow 0 < 24\mathrm{e} - 65 < 1$	M1
	Hence $\frac{65}{24} < \mathrm{e} < \frac{66}{24} \Rightarrow \frac{65}{24} < \mathrm{e} < \frac{11}{4}$	A1 (3) **[10]**

COMPLEX NUMBERS

Q1.

7 **(a)** Use de Moivre's theorem to show that

$$\operatorname{cosec} 7\theta = \frac{\operatorname{cosec}^7\theta}{7\operatorname{cosec}^6\theta - 56\operatorname{cosec}^4\theta + 112\operatorname{cosec}^2\theta - 64}.$$

[6]

(b) Hence obtain the roots of the equation

$$x^7 - 14x^6 + 112x^4 - 224x^2 + 128 = 0$$

in the form $\operatorname{cosec} q\pi$, where q is rational. [5]

Q2.

1 Find the roots of the equation $z^3 = 7\sqrt{3} - 7\mathrm{i}$, giving your answers in the form $re^{i\theta}$, where $r > 0$ and $-\pi \leqslant \theta < \pi$. [5]

5 **(a)** State the sum of the series $z+z^2+z^3+...+z^n$, for $z \neq 1$. [1]

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(b) Given that z is an nth root of unity and $z \neq 1$, deduce that $1+z+z^2+...+z^{n-1}=0$. [2]

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(c) Given instead that $z=\frac{1}{3}(\cos\theta+\mathrm{i}\sin\theta)$, use de Moivre's theorem to show that

$$\sum_{m=1}^{\infty} 3^{-m}\cos m\theta = \frac{3\cos\theta-1}{10-6\cos\theta}.$$ [7]

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Q4.

1 **(a)** Find a and b such that

$$z^8 - \mathrm{i}z^5 - z^3 + \mathrm{i} = (z^5 - a)(z^3 - b). \quad [1]$$

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(b) Hence find the roots of

$$z^8 - \mathrm{i}z^5 - z^3 + \mathrm{i} = 0,$$

giving your answers in the form $re^{\mathrm{i}\theta}$, where $r > 0$ and $0 \leqslant \theta < 2\pi$. [6]

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Q5.

4 By considering the binomial expansions of $\left(z+\frac{1}{z}\right)^5$ and $\left(z-\frac{1}{z}\right)^5$, where $z=\cos\theta+\mathrm{i}\sin\theta$, use de Moivre's theorem to show that

$$\tan^5\theta=\frac{\sin 5\theta-a\sin 3\theta+b\sin\theta}{\cos 5\theta+a\cos 3\theta+b\cos\theta},$$

where a and b are integers to be determined. [7]

Q6.

6 **(a)** Use de Moivre's theorem to show that

$$\operatorname{cosec} 5\theta = \frac{\operatorname{cosec}^5 \theta}{5\operatorname{cosec}^4 \theta - 20\operatorname{cosec}^2 \theta + 16}. \qquad [6]$$

(b) Hence obtain the roots of the equation

$$x^5 - 10x^4 + 40x^2 - 32 = 0$$

in the form $\operatorname{cosec}(q\pi)$, where q is rational. [4]

4 **(a)** Write down all the roots of the equation $x^5 - 1 = 0$. [2]

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(b) Use de Moivre's theorem to show that $\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$. [4]

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(c) Use the results of parts **(a)** and **(b)** to express each real root of the equation

$$8x^9 - 8x^7 + x^5 - 8x^4 + 8x^2 - 1 = 0$$

in the form $\cos k\pi$, where k is a rational number. [4]

Q8.

3 **(a)** Find the roots of the equation $z^3 = -1 - \mathrm{i}$, giving your answers in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r > 0$ and $0 \leqslant \theta < 2\pi$. [5]

Let $w = z_1^{3k} + z_2^{3k} + z_3^{3k}$, where k is a positive integer and z_1, z_2, z_3 are the roots of $z^3 = -1 - \mathrm{i}$.

(b) Express w in the form $R\mathrm{e}^{\mathrm{i}\alpha}$, where $R > 0$, giving R and α in terms of k. [3]

Q9.

8 **(a)** Use de Moivre's theorem to show that $\sin^6\theta = -\frac{1}{32}(\cos 6\theta - 6\cos 4\theta + 15\cos 2\theta - 10)$. [6]

It is given that $\cos^6\theta = \frac{1}{32}(\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10)$.

(b) Find the exact value of $\int_0^{\frac{1}{3}\pi}\left(\cos^6(\frac{1}{4}x) + \sin^6(\frac{1}{4}x)\right)\mathrm{d}x$. [4]

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(c) Express each root of the equation $16c^6 + 16\left(1 - c^2\right)^3 - 13 = 0$ in the form $\cos k\pi$, where k is a rational number. [5]

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Q10.

8 **(i)** Prove by mathematical induction that, for $z \neq 1$ and all positive integers n,

$$1 + z + z^2 + \ldots + z^{n-1} = \frac{z^n - 1}{z - 1}.$$ [5]

(ii) By letting $z = \frac{1}{2}(\cos\theta + \mathrm{i}\sin\theta)$, use de Moivre's theorem to deduce that

$$\sum_{m=1}^{\infty} \left(\tfrac{1}{2}\right)^m \sin m\theta = \frac{2\sin\theta}{5 - 4\cos\theta}. \qquad [5]$$

Q11.

9 **(i)** Use de Moivre's theorem to show that

$$\sec 6\theta = \frac{\sec^6 \theta}{32 - 48\sec^2 \theta + 18\sec^4 \theta - \sec^6 \theta}. \qquad [6]$$

(ii) Hence obtain the roots of the equation

$$3x^6 - 36x^4 + 96x^2 - 64 = 0$$

in the form $\sec q\pi$, where q is rational. [5]

Q12.

(i) Show that if $z = e^{i\theta}$ and $z \neq -1$ then

$$\frac{z-1}{z+1} = i \tan \tfrac{1}{2}\theta. \qquad [3]$$

(ii) Hence, or otherwise, show that if z is a cube root of unity then

$$\frac{z^3-1}{z^3+1}+\frac{z^2-1}{z^2+1}+\frac{z-1}{z+1}=0. \qquad [5]$$

(iii) Hence write down three roots of the equation

$$(z^3 - 1)(z^2 + 1)(z + 1) + (z^2 - 1)(z^3 + 1)(z + 1) + (z - 1)(z^3 + 1)(z^2 + 1) = 0$$

and find the other three roots. Give your answers in an exact form. [6]

Q13.

3 **(i)** Use de Moivre's theorem to show that

$$\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta. \qquad [3]$$

(ii) Hence find all the roots of the equation

$$x^4 - 6x^2 + 1 = 0$$

in the form $\tan q\pi$, where q is a positive rational number. [5]

Q14.

7 **(i)** Use de Moivre's theorem to show that

$$\sin 8\theta = 8 \sin\theta \cos\theta (1 - 10 \sin^2\theta + 24 \sin^4\theta - 16 \sin^6\theta). \quad [6]$$

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(ii) Use the equation $\dfrac{\sin 8\theta}{\sin 2\theta} = 0$ to find the roots of

$$16x^6 - 24x^4 + 10x^2 - 1 = 0$$

in the form $\sin k\pi$, where k is rational. [4]

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Q15.

8 **(i)** By considering the binomial expansion of $\left(z + \frac{1}{z}\right)^6$, where $z = \cos\theta + \mathrm{i}\sin\theta$, express $\cos^6\theta$ in the form

$$\tfrac{1}{32}(p + q\cos 2\theta + r\cos 4\theta + s\cos 6\theta),$$

where p, q, r and s are integers to be determined. [6]

(ii) Hence find the exact value of

$$\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \cos^6\left(\tfrac{1}{2}x\right) \mathrm{d}x. \qquad [4]$$

Q16.

8 **(i)** Let $z = \cos\theta + \mathrm{i}\sin\theta$. Show that $z - \frac{1}{z} = 2\mathrm{i}\sin\theta$ and hence express $16\sin^5\theta$ in the form $\sin 5\theta + p\sin 3\theta + q\sin\theta$, where p and q are integers to be determined. [6]

(ii) Hence find the exact value of $\int_0^{\frac{1}{3}\pi} 16\sin^5\theta\,\mathrm{d}\theta$. [3]

Q17.

7 **(i)** Use de Moivre's theorem to prove that

$$\tan 4\theta = \frac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}.$$ [5]

(ii) Hence find the solutions of the equation

$$t^4 - 4t^3 - 6t^2 + 4t + 1 = 0,$$

giving your answers in the form $\tan k\pi$, where k is a rational number. [5]

Q18.

10 **(i)** Use de Moivre's theorem to show that

$$\sin 5\theta = 5\sin\theta - 20\sin^3\theta + 16\sin^5\theta.$$ [5]

(ii) Hence explain why the roots of the equation $16x^4 - 20x^2 + 5 = 0$ are $x = \pm \sin \frac{1}{5}\pi$ and $x = \pm \sin \frac{2}{5}\pi$. [3]

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(iii) Without using a calculator, find the exact values of

$$\sin \tfrac{1}{5}\pi \sin \tfrac{2}{5}\pi \sin \tfrac{3}{5}\pi \sin \tfrac{4}{5}\pi \quad \text{and} \quad \sin^2\left(\tfrac{1}{5}\pi\right) + \sin^2\left(\tfrac{2}{5}\pi\right). \qquad [4]$$

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Q19.

6 Use de Moivre's theorem to express $\cot 7\theta$ in terms of $\cot \theta$. [4]

Use the equation $\cot 7\theta = 0$ to show that the roots of the equation

$$x^6 - 21x^4 + 35x^2 - 7 = 0$$

are $\cot\left(\frac{1}{14}k\pi\right)$ for $k = 1, 3, 5, 9, 11, 13$, and deduce that

$$\cot^2\left(\tfrac{1}{14}\pi\right)\cot^2\left(\tfrac{3}{14}\pi\right)\cot^2\left(\tfrac{5}{14}\pi\right) = 7.$$ [5]

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Q20.

9 Use de Moivre's theorem to show that $\cos^4\theta = \frac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3)$. [4]

Find the corresponding expression for $\sin^4\theta$ in terms of $\cos 4\theta$ and $\cos 2\theta$. [4]

Hence find the exact value of $\int_0^{\frac{1}{8}\pi} (\cos^4\theta + \sin^4\theta)\,\mathrm{d}\theta$. [3]

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Q21.

10 Let $z = \cos\theta + \mathrm{i}\sin\theta$. Show that

$$z^n + \frac{1}{z^n} = 2\cos n\theta \qquad \text{and} \qquad z^n - \frac{1}{z^n} = 2\mathrm{i}\sin n\theta. \qquad [2]$$

By considering $\left(z - \frac{1}{z}\right)^4 \left(z + \frac{1}{z}\right)^2$, show that

$$\sin^4\theta\cos^2\theta = \tfrac{1}{32}(\cos 6\theta - 2\cos 4\theta - \cos 2\theta + 2). \qquad [7]$$

Hence find the exact value of $\int_0^{\frac{1}{4}\pi} \sin^4\theta\cos^2\theta\,\mathrm{d}\theta$. [3]

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Q22.

8 By considering $\sum_{r=1}^{n} z^{2r-1}$, where $z = \cos\theta + \mathrm{i}\sin\theta$, show that, if $\sin\theta \neq 0$,

$$\sum_{r=1}^{n} \sin(2r-1)\theta = \frac{\sin^2 n\theta}{\sin\theta}. \qquad [7]$$

Deduce that

$$\sum_{r=1}^{n} (2r-1)\cos\left[\frac{(2r-1)\pi}{2n}\right] = -\operatorname{cosec}\left(\frac{\pi}{2n}\right)\cot\left(\frac{\pi}{2n}\right). \qquad [4]$$

Q23.

6 Let $z = \cos\theta + \mathrm{i}\sin\theta$. Use the binomial expansion of $(1+z)^n$, where n is a positive integer, to show that

$$\binom{n}{1}\cos\theta + \binom{n}{2}\cos 2\theta + \ldots + \binom{n}{n}\cos n\theta = 2^n \cos^n\left(\tfrac{1}{2}\theta\right)\cos\left(\tfrac{1}{2}n\theta\right) - 1. \quad [7]$$

Find

$$\binom{n}{1}\sin\theta + \binom{n}{2}\sin 2\theta + \ldots + \binom{n}{n}\sin n\theta. \quad [2]$$

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Q24.

10 Using de Moivre's theorem, show that

$$\tan 5\theta = \frac{5\tan\theta - 10\tan^3\theta + \tan^5\theta}{1 - 10\tan^2\theta + 5\tan^4\theta}. \qquad [5]$$

Hence show that the equation $x^2 - 10x + 5 = 0$ has roots $\tan^2\left(\frac{1}{5}\pi\right)$ and $\tan^2\left(\frac{2}{5}\pi\right)$. [4]

Deduce a quadratic equation, with integer coefficients, having roots $\sec^2\left(\frac{1}{5}\pi\right)$ and $\sec^2\left(\frac{2}{5}\pi\right)$. [3]

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Q25.

7 Use de Moivre's theorem to show that

$$\tan 5\theta = \frac{5t - 10t^3 + t^5}{1 - 10t^2 + 5t^4},$$

where $t = \tan\theta$. [4]

Deduce that the roots of the equation $t^4 - 10t^2 + 5 = 0$ are $\pm\tan\frac{1}{5}\pi$ and $\pm\tan\frac{2}{5}\pi$. [3]

Hence show that $\tan\frac{1}{5}\pi\tan\frac{2}{5}\pi = \sqrt{5}$. [2]

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Q26.

5 State the sum of the series $z + z^2 + z^3 + \ldots + z^n$, for $z \neq 1$. [1]

By letting $z = \cos\theta + \mathrm{i}\sin\theta$, show that

$$\cos\theta + \cos 2\theta + \cos 3\theta + \ldots + \cos n\theta = \frac{\sin \frac{1}{2}n\theta}{\sin \frac{1}{2}\theta} \cos \tfrac{1}{2}(n+1)\theta,$$

where $\sin \frac{1}{2}\theta \neq 0$. [7]

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Q27.

6 Use de Moivre's theorem to show that

$$\cos 5\theta \equiv \cos\theta(16\sin^4\theta - 12\sin^2\theta + 1). \qquad [5]$$

By considering the equation $\cos 5\theta = 0$, show that the exact value of $\sin^2\left(\frac{1}{10}\pi\right)$ is $\frac{3-\sqrt{5}}{8}$. [4]

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COMPEX NUMBERS
MS

Q1.

Question	Answer	Marks	Guidance
7(a)	$\sin 7\theta = \text{Im}(c+\text{i}s)^7$	**M1**	Uses binomial theorem.
	$\sin 7\theta = -s^7 + 21c^2s^5 - 35c^4s^3 + 7c^6s$	**A1**	
	$\sin 7\theta = -s^7 + 21(1-s^2)s^5 - 35(1-s^2)^2s^3 + 7(1-s^2)^3s$ $\sin 7\theta = -s^7 + 21(1-s^2)s^5 - 35(1-2s^2+s^4)s^3 + 7(1-3s^2+3s^4-s^6)s$	**M1**	Uses $c^2 = 1-s^2$.
	$\sin 7\theta = -64s^7 + 112s^5 - 56s^3 + 7s$	**A1**	
	$\text{cosec}\, 7\theta = \dfrac{1}{-64s^7 + 112s^5 - 56s^3 + 7s}$	**M1**	Divides numerator and denominator by s^7.
	$\text{cosec}\, 7\theta = \dfrac{\text{cosec}^7\theta}{7\text{cosec}^6\theta - 56\text{cosec}^4\theta + 112\text{cosec}^2\theta - 64}$	**A1**	AG CWO
		6	
7(b)	$x^7 = 2(7x^6 - 56x^4 + 112x^2 - 64)$ leading to $\dfrac{x^7}{7x^6 - 56x^4 + 112x^2 - 64} = 2$	**M1 A1**	Relates with equation in part (a).
	$\text{cosec}\, 7\theta = 2$ leading to $\sin 7\theta = \frac{1}{2}$	**M1**	Solves $\sin 7\theta = \frac{1}{2}$.
	$x = \text{cosec}\left(\frac{1}{42}\pi\right)$	**A1**	Gives one correct solution.
	$x = \text{cosec}(q\pi)$, $q = -\frac{19}{42}, -\frac{11}{42}, -\frac{7}{42}, \frac{5}{42}, \frac{13}{42}, \frac{17}{42}$	**A1**	Gives six other solutions. Allow different values of q as long as all seven solutions are found.
		5	

Q2.

Question	Answer	Marks	Guidance
1	$z^3 = 7\sqrt{3} - 7\text{i} = 14\text{e}^{-\text{i}\frac{1}{6}\pi}$	**B1**	Converts $7\sqrt{3} - 7\text{i}$ to exponential form.
	$z_1 = 14^{\frac{1}{3}}\text{e}^{-\text{i}\frac{1}{18}\pi}$	**M1 A1**	Finds one root.
	$z_2 = 14^{\frac{1}{3}}\text{e}^{\text{i}\frac{11}{18}\pi}$, $z_3 = 14^{\frac{1}{3}}\text{e}^{-\text{i}\frac{13}{18}\pi}$	**A1FT A1FT**	Finds other two roots. FT on modulus. Others given scores A1 A0.
		5	

Question	Answer	Marks	Guidance
5(a)	$\frac{z-z^{n+1}}{1-z}$ or $\frac{z^{n+1}-z}{z-1}$.	**B1**	
		1	
5(b)	$z^n=1$ and $z\neq 1$ leading to $z+z^2+z^3+...+z^n=0$ leading to $1+z+z^2+...+z^{n-1}=0$	**M1 A1**	Must see $z^n=1$.
		2	
5(c)	$\sum_{m=1}^{\infty} z^m=\frac{z}{1-z}=\frac{\cos\theta+\mathrm{i}\sin\theta}{3-\cos\theta-\mathrm{i}\sin\theta}$	**M1 A1**	Applies sum to infinity and substitutes for z.
	$\frac{(\cos\theta+\mathrm{i}\sin\theta)(3-\cos\theta+\mathrm{i}\sin\theta)}{(3-\cos\theta)^2+\sin^2\theta}$	**M1 A1**	Rationalises denominator.
	$\frac{3\cos\theta-\cos^2\theta-\sin^2\theta+\mathrm{i}\sin\theta\cos\theta+\mathrm{i}\sin\theta(3-\cos\theta)}{9-6\cos\theta+\cos^2\theta+\sin^2\theta}$	**M1**	Applies $\sin^2\theta+\cos^2\theta=1$ or $\mathrm{e}^{\mathrm{i}\theta}+\mathrm{e}^{-\mathrm{i}\theta}=2\cos\theta$.
	$\mathrm{Re}\left(\sum_{m=1}^{\infty} z^m\right)=\frac{3\cos\theta-1}{10-6\cos\theta}$	**M1 A1**	Takes the real part, AG.
		7	

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Question	Answer	Marks	Guidance
1(a)	$a = 1, \quad b = \mathrm{i}$	**B1**	
		1	
1(b)	$z = \mathrm{e}^{\mathrm{i}\frac{2k}{5}\pi}$	**M1**	Finds one fifth root of unity.
	$z = \mathrm{e}^{\mathrm{i}\frac{2k}{5}\pi}, \quad k = 0,1,2,3,4$	**A1**	Gives all fifth roots of unity. Accept 1 not in exponential form. A0 if $r = 1$ not seen or implied.
	Argument of i is $\frac{1}{2}\pi$.	**B1**	
	$z = \mathrm{e}^{\mathrm{i}\frac{1}{6}\pi}$	**M1 A1**	Finds one root of $z^3 = \mathrm{i}$. A0 if first root not given in exponential form.
	$z = \mathrm{e}^{\mathrm{i}\frac{5}{6}\pi}, \quad \mathrm{e}^{\mathrm{i}\frac{3}{2}\pi}$	**A1**	Finds other two roots of $z^3 = \mathrm{i}$. A0 if $r = 1$ not seen or implied. Withhold final A1 if a and b reversed in part **(b)** but all other work correct.
		6	**SC** If $z = \mathrm{e}^{\mathrm{i}\left(\frac{1}{10}\pi + \frac{2k}{5}\pi\right)}, \quad k = 0,1,2,3,4$ award M1 A1 A1 only. **SC** If $z = \mathrm{e}^{\mathrm{i}\left(\frac{2k}{3}\pi\right)}, \quad k = 0,1,2$ award M1 A1 only.

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Question	Answer	Marks	Guidance
4	$z - z^{-1} = 2\mathrm{i}\sin\theta$ and $z + z^{-1} = 2\cos\theta$	**B1**	Use of $z - z^{-1} = 2\mathrm{i}\sin\theta$ and $z + z^{-1} = 2\cos\theta$
	$\left(z - z^{-1}\right)^5 = \left(z^5 - z^{-5}\right) - 5\left(z^3 - z^{-3}\right) + 10\left(z - z^{-1}\right)$	**M1 A1**	Expands and groups. A0 if grouping not shown.
	$\left(z + z^{-1}\right)^5 = \left(z^5 + z^{-5}\right) + 5\left(z^3 + z^{-3}\right) + 10\left(z + z^{-1}\right)$	**M1 A1**	Expands and groups. A0 if grouping not shown.
	$\dfrac{2^5\mathrm{i}\sin^5\theta}{2^5\cos^5\theta} = \dfrac{2\mathrm{i}\sin 5\theta - 5(2\mathrm{i}\sin 3\theta) + 10(2\mathrm{i}\sin\theta)}{2\cos 5\theta + 5\left(2\cos 3\theta\right) + 10\left(2\cos\theta\right)}$	**M1**	Substitutes $z^n + z^{-n} = 2\cos n\theta$ and $z^n - z^{-n} = 2\mathrm{i}\sin n\theta$.
	$\tan^5\theta = \dfrac{\sin 5\theta - 5\sin 3\theta + 10\sin\theta}{\cos 5\theta + 5\cos 3\theta + 10\cos\theta}$.	**A1**	Justifies cancellation of constants.
		7	

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Question	Answer	Marks	Guidance
6(a)	$(\cos 5\theta + \mathrm{i}\sin 5\theta) = (c + \mathrm{i}s)^5$	**M1**	Uses binomial theorem.
	$\sin 5\theta = s^5 - 10c^2s^3 + 5c^4s$	**A1**	
	$s^5 - 10(1-s^2)s^3 + 5(1-s^2)^2 s$	**M1**	Uses $c^2 = 1 - s^2$ or $\cot^2\theta = \mathrm{cosec}^2\theta - 1$ after dividing numerator and denominator by s^5.
	$16s^5 - 20s^3 + 5s$	**A1**	
	$\mathrm{cosec}\,5\theta = \dfrac{1}{16s^5 - 20s^3 + 5s} \times \dfrac{\mathrm{cosec}^5\theta}{\mathrm{cosec}^5\theta}$	**M1**	Divides *simplified* numerator and denominator by s^5.
	$\dfrac{\mathrm{cosec}^5\theta}{5\mathrm{cosec}^4\theta - 20\mathrm{cosec}^2\theta + 16}$	**A1**	AG
		6	
6(b)	$x^5 = 2(5x^4 - 20x^2 + 16)$ leading to $\dfrac{x^5}{5x^4 - 20x^2 + 16} = 2$	**M1**	Relates with equation in part (a).
	$\mathrm{cosec}\,5\theta = 2$ leading to $\sin 5\theta = \frac{1}{2}$	**M1**	Solves $\sin 5\theta = \frac{1}{2}$.
	$x = \mathrm{cosec}\left(\frac{1}{30}\pi\right)$	**A1**	Gives one correct solution.
	$\mathrm{cosec}\left(\frac{5}{30}\pi\right), \mathrm{cosec}\left(-\frac{7}{30}\pi\right), \mathrm{cosec}\left(-\frac{11}{30}\pi\right), \mathrm{cosec}\left(\frac{13}{30}\pi\right)$	**A1**	Gives four other solutions. Allow different values of q as long as all five solutions are found. 9.56, 2, –1.49, –1.09, 1.02
		4	

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Q7.

Question	Answer	Marks	Guidance
4(a)	$e^{i\frac{2}{5}k\pi}, \quad k = 0,1,2,3,4$	**B2**	OE. B1 for 1 correct fifth root of unity. B2 for exactly 5 distinct, correct roots.
		2	
4(b)	$(c+is)^4 = c^4 + 4c^3(is) + 6c^2(is)^2 + 4c(is)^3 + (is)^4$	**M1 A1**	Uses binomial expansion. Can also go RHS to LHS
		A1	AG. Can get final A1 without first. **SC:** Using double angle formulae scores 1/4.
		4	

Question	Answer	Marks	Guidance
4(c)	$8x^9 - 8x^7 + x^5 - 8x^4 + 8x^2 - 1 = (x^5 - 1)(8x^4 - 8x^2 + 1)$	**B1**	
	$\overline{2}$		$4\theta = \frac{1}{2}\pi, \frac{3}{2}\pi, \frac{5}{2}\pi, \frac{7}{2}\pi$ OE.
	$\cos 0, \cos\frac{1}{8}\pi, \cos\frac{3}{8}\pi, \cos\frac{5}{8}\pi, \cos\frac{7}{8}\pi$	**A1**	Gives exactly 5 distinct, real roots. Accept $1, \pm\cos\frac{1}{8}\pi, \pm\cos\frac{3}{8}\pi$.
		4	

Q8.

Question	Answer	Marks
3(a)	$z^3 = -1 - \mathrm{i} = 2^{\frac{1}{2}} \mathrm{e}^{\mathrm{i}\frac{5}{4}\pi}$	**B1**
	$z_1 = 2^{\frac{1}{6}} \mathrm{e}^{\mathrm{i}\frac{5}{12}\pi}$	**M1 A1**
	$z_2 = 2^{\frac{1}{6}} \mathrm{e}^{\mathrm{i}\frac{13}{12}\pi}$, $z_3 = 2^{\frac{1}{6}} \mathrm{e}^{\mathrm{i}\frac{21}{12}\pi}$	**A1 A1**
		5
3(b)	$z_1^{3k} + z_2^{3k} + z_3^{3k} = 3\left(2^{\frac{1}{2}k} \mathrm{e}^{\mathrm{i}\frac{5}{4}k\pi}\right)$	**M1**
	$R = \mid w \mid = 3\left(2^{\frac{1}{2}k}\right)$	**A1**
	$\alpha = \frac{5}{4}k\pi$	**A1**
		3

Question	Answer	Marks
8(a)	$z - z^{-1} = 2\mathrm{i}\sin\theta$	**B1**
	$\left(z-z^{-1}\right)^6 = \left(z^6+z^{-6}\right)-6\left(z^4+z^{-4}\right)+15\left(z^2+z^{-2}\right)-20$	**M1 A1**
	$\left(2\mathrm{i}\sin\theta\right)^6 = 2\cos 6\theta - 6\left(2\cos 4\theta\right)+15\left(2\cos 2\theta\right)-20$	**M1 A1**
	$\sin^6\theta = \frac{1}{32}\left(-\cos 6\theta+6\cos 4\theta-15\cos 2\theta+10\right)$	**A1**
		6
8(b)	$\int_0^{\frac{1}{3}\pi}\cos^6\tfrac{1}{4}x+\sin^6\tfrac{1}{4}x\,\mathrm{d}x = \tfrac{1}{8}\int_0^{\frac{1}{3}\pi}3\cos x+5\,\mathrm{d}x$	**M1 A1**
	$\frac{1}{8}\left[3\sin x+5x\right]_0^{\frac{1}{3}\pi} = \frac{1}{8}\left(\frac{5}{3}\pi+\frac{3}{2}\sqrt{3}\right).$	**M1 A1**
		4
8(c)	$c = \cos\theta \Rightarrow 1-c^2 = \sin^2\theta$	**B1**
	$16c^6+16\left(1-c^2\right)^3-13=0 \Rightarrow 6\cos 4\theta-3=0$	**M1 A1**
	$4\theta = \frac{1}{3}\pi, \frac{5}{3}\pi, \frac{7}{3}\pi, \frac{11}{3}\pi$	**M1**
	$c = \cos\left(\frac{1}{12}\pi\right), \cos\left(\frac{5}{12}\pi\right), \cos\left(\frac{7}{12}\pi\right), \cos\left(\frac{11}{12}\pi\right)$	**A1**
		5

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Question	Answer	Marks	Guidance
8(i)	$1=\dfrac{z^1-1}{z-1}$ So true when $n=1$.	**B1**	Shows base case.
	Assume that $1+z+\ldots+z^{k-1}=\dfrac{z^k-1}{z-1}$	**B1**	States inductive hypothesis.
	Then $1+z+\ldots+z^{k-1}+z^k=\dfrac{z^k-1}{z-1}+z^k=\dfrac{z^k-1+z^k(z-1)}{z-1}=\dfrac{z^{k+1}-1}{z-1}$, so true when $n=k+1$	**M1 A1**	Combines fractions.
	$H_k \rightarrow H_{k+1}$ Hence, by induction, true for all positive integers.	**A1**	States conclusion.
		5	

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Question	Answer	Marks	Guidance
8(ii)	Since $\lvert z\rvert<1$, $\sum_{m=0}^{\infty} z^m = \frac{-1}{z-1}$	**B1**	States $\lvert z\rvert<1$ and uses formula for sum to infinity of geometric progression.
	$\sum_{m=1}^{\infty} 2^{-m}\sin m\theta = \text{Im}\left(\sum_{m=0}^{\infty} z^m\right) = \text{Im}\left(\frac{-1}{\frac{1}{2}\cos\theta + \text{i}\frac{1}{2}\sin\theta - 1}\right)$	**M1 A1**	Uses de Moivre's theorem.
	$\text{Im}\left(\frac{-\left(\frac{1}{2}\cos\theta - 1 - \text{i}\frac{1}{2}\sin\theta\right)}{\frac{1}{4}\cos^2\theta - \cos\theta + 1 + \frac{1}{4}\sin^2\theta}\right)$	**M1**	Multiply numerator and denominator by conjugate.
	$\frac{\frac{1}{2}\sin\theta}{\frac{5}{4}-\cos\theta} = \frac{2\sin\theta}{5-4\cos\theta}$	**A1**	States imaginary part, AG.
		5	

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Q11.

Question	Answer	Marks	Guidance
9(i)	Write $c=\cos\theta,\ s=\sin\theta.$ $)^6$	**M1**	Uses binomial theorem.
		A1	
	$c^6-15c^4s^2+15c^2s^4-s^6=c^6-15c^4\left(1-c^2\right)+15c^2\left(1-c^2\right)^2-\left(1-c^2\right)^3$	**M1**	Uses $c^2=1-s^2$.
	$=c^6-15c^4\left(1-c^2\right)+15c^2\left(1-2c^2+c^4\right)-\left(1-3c^2+3c^4-c^6\right)$	**A1**	
	$=32c^6-48c^4+18c^2-1$	**M1**	Divides numerator and denominator by c^6.
	$\Rightarrow \sec 6\theta=\dfrac{1}{32c^6-48c^4+18c^2-1}=\dfrac{\sec^6\theta}{32-48\sec^2\theta+18\sec^4\theta-\sec^6\theta}$	**A1**	AG
		6	

Question	Answer	Marks	Guidance
9(ii)	$x^6=2\left(32-48x^2+18x^4-x^6\right)\Rightarrow \dfrac{x^6}{32-48x^2+18x^4-x^6}=2$	**M1 A1**	Relates with equation in part (i).
	$\sec 6\theta=2\Rightarrow\cos 6\theta=\dfrac{1}{2}$	**M1**	Solves $\cos 6\theta=\dfrac{1}{2}$.
	$x=\sec\dfrac{\pi}{18}$	**A1**	Gives one correct solution.
	$x=\sec q\pi,\ \ q=\dfrac{5}{18},\dfrac{7}{18},\dfrac{11}{18},\dfrac{13}{18},\dfrac{17}{18}$	**A1**	Gives five other solutions. Allow different values of q as long as all six solutions are found.
		5	

Question	Answer	Marks	Guidance
11E(i)	If $z \neq -1$ then $\dfrac{z-1}{z+1} = \dfrac{z^{\frac{1}{2}} - z^{-\frac{1}{2}}}{z^{\frac{1}{2}} + z^{-\frac{1}{2}}} \left(\text{or} = \dfrac{2i\sin\theta}{2(\cos\theta+1)} \right)$	**M1 A1**	Using de Moivre's theorem or multiplying numerator and denominator by $\cos\theta + 1 - i\sin\theta$
	$= \dfrac{2i\sin\frac{1}{2}\theta}{2\cos\frac{1}{2}\theta} = i\tan\frac{1}{2}\theta$	**A1**	
		3	
11E(ii)	If $z = 1$ then $z^r - 1 = 0$ so the given sum is zero.	**B1**	
	If $z \neq 1$ then $z = e^{i\frac{2\pi}{3}}$ or $z = e^{-i\frac{2\pi}{3}}$, so	**B1**	Must consider all three roots of unity
	$\dfrac{z^3-1}{z^3+1} + \dfrac{z^2-1}{z^2+1} + \dfrac{z-1}{z+1} = 0 + \dfrac{e^{\pm i\frac{4\pi}{3}} - 1}{e^{\pm i\frac{4\pi}{3}} + 1} + \dfrac{e^{\pm i\frac{2\pi}{3}} - 1}{e^{\pm i\frac{2\pi}{3}} + 1}$	**M1 A1**	AEF two of three terms correct for M1
	Using **(ii)**, or by direct calculation, $\dfrac{e^{\pm i\frac{4\pi}{3}} - 1}{e^{\pm i\frac{4\pi}{3}} + 1} + \dfrac{e^{\pm i\frac{2\pi}{3}} - 1}{e^{\pm i\frac{2\pi}{3}} + 1} = i\tan\frac{1}{2}\left(\frac{4\pi}{3}\right) + i\tan\frac{1}{2}\left(\frac{2\pi}{3}\right) = 0$	**A1**	AG
		5	

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Question	Answer	Marks	Guidance
11E(iii)	$z=1,\ e^{\pm i\frac{2\pi}{3}}$, $\left(z^3-1\right)\left(z^2+1\right)(z+1)+\left(z^2-1\right)\left(z^3+1\right)(z+1)+(z-1)\left(z^3+1\right)\left(z^2+1\right)=0$	**B1**	Writes down the cube roots of unity. AEF, must be exact
	$\Rightarrow 3z^6+z^5+z^4-z^2-z-3=0$	**M1**	Expands
	$3z^6+z^5+z^4-z^2-z-3=(z+1)\left(z^3-1\right)\left(3z^2-2z+3\right)$	**M1 A1**	Factorises
	$\Rightarrow z=-1,\ \ \frac{2\pm i\sqrt{32}}{6}=\frac{1}{3}\pm i\frac{2}{3}\sqrt{2}$	**M1 A1**	Finds other three roots. AEF, must be exact
		6	

Q13.

May/June 2018

Question	Answer	Marks	Guidance
3(i)	$(c+is)^4=c^4+4c^3(is)+6c^2\left(-s^2\right)+4c(is)^3+(is)^4$	**M1**	Uses binomial theorem to expand $(c+is)^4$.
	$\Rightarrow \cos 4\theta=c^4-6c^2s^2+s^4$	**M1 A1**	Takes real part, AG.
3(ii)	$\frac{\cos 4\theta}{\cos^4\theta}=\tan^4\theta-6\tan^2\theta+1$	**M1 A1**	Divides through by $\cos^4\theta$
	Let $x=\tan\theta, then\ x^4-6x^2+1=0\Rightarrow\cos 4\theta=0$ $\Rightarrow 4\theta=\pm\frac{\pi}{2}+2m\pi,\ m\in\mathbb{Z}$	**dM1**	Following on from finding correct quartic Solves $\cos 4\theta=0$.
	Roots are $\tan q\pi$ where $q=\frac{1}{8},\frac{3}{8},\frac{5}{8},\frac{7}{8}$. (q > 0)	**A1 A1**	**Alt methods:** Solves $\tan(4\theta)=(4t-t^3)/(t^4-6t^2+1)$ and $\tan(4\theta)=\infty$. Solves equation in $\cot(\theta)$ after dividing by $\sin^4\theta$
		8	

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Question	Answer	Marks	Guidance
7(i)	Write $c=\cos\theta$, $s=\sin\theta$. $\cos 8\theta + i\sin 8\theta = (c+is)^8$	**M1**	Uses binomial theorem.
	$\Rightarrow \sin 8\theta = 8c^7 s - 56c^5 s^3 + 56c^3 s^5 - 8cs^7$	**A1**	
	$= 8cs\left(c^6 - 7c^4 s^2 + 7c^2 s^4 - s^6\right)$	**M1**	Factorises.
	$c^6 - 7c^4 s^2 + 7c^2 s^4 - s^6 = \left(1-s^2\right)^3 - 7\left(1-s^2\right)^2 s^2 + 7\left(1-s^2\right)s^4 - s^6$	**M1**	Uses $c^2 = 1-s^2$.
	$= \left(1-3s^2+3s^4-s^6\right) - 7\left(s^2-2s^4+s^6\right) + 7\left(s^4-s^6\right) - s^6$ $= 1-10s^2+24s^4-16s^6$	**A1**	
	$\Rightarrow \sin 8\theta = 8cs\left(1-10s^2+24s^4-16s^6\right)$	**A1**	AG
		6	

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Question	Answer	Marks	Guidance
7(ii)	$\sin 2\theta = 2cs \Rightarrow \dfrac{\sin 8\theta}{\sin 2\theta} = 4\left(1-10s^2+24s^4-16s^6\right)$	**M1**	Uses $\sin 2\theta = 2cs$ to relate with equation in part (i)
	$\sin 8\theta = 0 \Rightarrow \theta = \dfrac{n\pi}{8}$	**M1**	Solves $\sin 8\theta = 0$
	$x = \sin\dfrac{\pi}{8}$	**A1**	Gives one correct solution
	$\sin\dfrac{n\pi}{8}$, $n = \pm 1, \pm 2 \pm 3 \, or \, 1,2,3,9,10,11$	**A1**	Gives five other solutions. Allow different values of k as long as six distinct solutions are found. $\sin 0$ and $\sin\dfrac{\pi}{2}$ must be excluded.
		4	

Q15.

Question	Answer	Marks	Guidance
8(i)			
	$\left(z+z^{-1}\right)^6 = \left(z^6+z^{-6}\right)+6\left(z^4+z^{-4}\right)+15\left(z^2+z^{-2}\right)+20$	**M1A1**	Expands and groups.
		M1A1	Substitutes $z^n + z^{-n} = 2\cos n\theta$.
	$\Rightarrow \cos^6\theta = \dfrac{1}{32}(10+15\cos 2\theta+6\cos 4\theta+\cos 6\theta)$	**A1**	(Allow $p=10$, $q=15, r=6, s=1$.)
		6	

Question	Answer	Marks	Guidance
8(ii)	$\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \cos^6\frac{x}{2}\,\mathrm{d}x = \frac{1}{32}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} 10+15\cos x+6\cos 2x+\cos 3x\,\mathrm{d}x$	**M1**	Applies part **(i)**
		M1	Integrates correctly (3/4 terms correct).
	$\frac{1}{32}\left[10x+15\sin x+3\sin 2x+\frac{1}{3}\sin 3x\right]_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}$	**M1**	Inserts limits and evaluates.
	$=\frac{1}{32}\left\{\left(5\pi+15+0-\frac{1}{3}\right)-\left(-5\pi-15+0+\frac{1}{3}\right)\right\}=\frac{1}{16}\left(5\pi+\frac{44}{3}\right)$	**A1**	
		4	

Q16.

8(i)	$z - z^{-1} = \cos\theta + \mathrm{i}\sin\theta - \cos(-\theta) - \mathrm{i}\sin(-\theta) = 2\mathrm{i}\sin\theta$	**B1**	
	$\left(z - \frac{1}{z}\right)^5 = \left(z^5 - \frac{1}{z^5}\right) - 5\left(z^3 - \frac{1}{z^3}\right) + 10\left(z - \frac{1}{z}\right)$	**M1A1**	
	$\Rightarrow 32\sin^5\theta\mathrm{i} = 2\mathrm{i}\sin 5\theta - 10\mathrm{i}\sin 3\theta + 20\mathrm{i}\sin\theta$	**M1A1**	Grouping not required at this stage.
	$\Rightarrow 16\sin^5\theta = \sin 5\theta - 5\sin 3\theta + 10\sin\theta$	**A1**	**M1** for grouping and applying initial result
	Total:	**6**	

Question	Answer	Marks	Guidance
8(ii)	$\int_0^{\frac{1}{3}\pi} 16\sin^5\theta \,\mathrm{d}\theta = \left[-\frac{\cos 5\theta}{5} + \frac{5\cos 3\theta}{3} - 10\cos\theta\right]_0^{\frac{1}{3}\pi}$	**M1A1FT**	
	$= \left[-\frac{1}{10} - \frac{5}{3} - 5\right] - \left[-\frac{1}{5} + \frac{5}{3} - 10\right] = \frac{53}{30}$	**A1**	
	Total:	**3**	

Q17.

Question	Answer	Marks	Guidance
7(i)	$(c+\mathrm{i}s)^4 = c^4 + 4c^3(\mathrm{i}s) + 6c^2(\mathrm{i}s)^2 + 4c(\mathrm{i}s)^3 + (\mathrm{i}s)^4$	**B1**	SOI
	Equate real and imaginary parts	**M1**	
	$\cos 4\theta = c^4 - 6c^2s^2 + s^4$	**A1**	SOI
	$\sin 4\theta = 4c^3s - 4cs^3$	**A1**	SOI
	$\tan 4\theta = \dfrac{(4c^3s - 4cs^3) \div c^4}{(c^4 - 6c^2s^2 + s^4) \div c^4} \left(= \dfrac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta} \right)$	**A1**	AG
	Total:	**5**	
7(ii)	$\tan 4\theta = -1 \Rightarrow t^4 - 4t^3 - 6t^2 + 4t + 1 = 0$	**M1 M1**	
	so $4\theta = \dfrac{3\pi}{4} \left(, \dfrac{7\pi}{4}, \dfrac{11\pi}{4}, \dfrac{15\pi}{4} \right)$	**M1**	
	$t = \tan\dfrac{3\pi}{16}, \tan\dfrac{7\pi}{16}, \tan\dfrac{11\pi}{16}, \tan\dfrac{15\pi}{16}.$ Allow $(\dfrac{k}{4} - \dfrac{1}{16})\pi$ $k = 0,1,2,3$ oe	**A1 A1**	
	Total:	**5**	

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Q18.

Question	Answer	Marks	Guidance
10(i)	$\sin 5\theta = Im(c+\mathrm{i}s)^5 =$ $Im\left(c^5 + 5c^4\mathrm{i}s + 10c^3(\mathrm{i}s)^2 + 10c^2(\mathrm{i}s)^3 + 5c(\mathrm{i}s)^4 + (\mathrm{i}s)^5\right)$	**B1**	SOI
	$\sin 5\theta = 5c^4 s - 10c^2 s^3 + s^5$	**M1A1**	
	$= s\left(5\left[1-s^2\right]^2 - 10s^2\left[1-s^2\right] + s^4\right)$	**M1**	
		A1	AG
		5	

Question	Answer	Marks	Guidance
10(ii)	If $\theta = 0, \pm\frac{1}{5}\pi, \pm\frac{2}{5}\pi$ then $\sin 5\theta = 0$	**B1**	
	$\Rightarrow 16s^5 - 20s^3 + 5s = 0$, where $s = \sin\theta$, $\Rightarrow s\left(16s^4 - 20s^2 + 5\right) = 0$	**B1**	
	$s = 0 \Rightarrow \theta = 0$	**B1**	
	Hence roots of $16s^4 - 20s^3 + 5 = 0$ are $\pm\sin\frac{1}{5}\pi, \pm\sin\frac{2}{5}\pi$		AG
		3	
10(iii)	Since $\sin\frac{4}{5}\pi = -\sin\left(-\frac{1}{5}\pi\right)$ and $\sin\frac{3}{5}\pi = -\sin\left(-\frac{2}{5}\pi\right)$	**B1**	
	$\sin\left(\frac{4}{5}\pi\right)\sin\left(\frac{3}{5}\pi\right)\sin\left(\frac{2}{5}\pi\right)\sin\left(\frac{1}{5}\pi\right) =$ $\sin\left(-\frac{1}{5}\pi\right)\sin\left(-\frac{2}{5}\pi\right)\sin\left(\frac{1}{5}\pi\right)\sin\left(\frac{2}{5}\pi\right) = \frac{5}{16}$	**M1A1**	
	$\sin^2\frac{1}{5}\pi + \sin^2\frac{2}{5}\pi = -\frac{(-20)}{16} = \frac{5}{4}$	**A1**	
		4	

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Qu	Soluti	Part Marks
6	$(c+\mathrm{i}s)^7 = c^7 + 7c^6(\mathrm{i}s) + \ldots + (\mathrm{i}s)^7$	**M1**
	$\dfrac{\cos 7\theta}{\sin 7\theta} = \dfrac{c^7 - 21c^5s^2 + 35c^3s^4 - 7cs^6}{7c^6s - 35c^4s^3 + 21c^2s^5 - s^7}$	**A1**
	$\cot 7\theta = \dfrac{\cot^7\theta - 21\cot^5\theta + 35\cot^3\theta - 7\cot\theta}{7\cot^6\theta - 35\cot^4\theta + 21\cot^2\theta - 1}$	**M1** **A1** [4]
	$\cot 7\theta = 0$ **and** $\cot\theta \neq 0 \Rightarrow x^6 - 21x^4 + 35x^2 - 7 = 0$, where $x = \cot\theta$	
	and $\theta = k\dfrac{\pi}{14}$ where $k = 1, 3, 5, 9, 11, 13$	**M1**
	Product of roots $\Rightarrow \cot\dfrac{\pi}{14}\cot\dfrac{3\pi}{14}\cot\dfrac{5\pi}{14}\cot\dfrac{9\pi}{14}\cot\dfrac{11\pi}{14}\cot\dfrac{13\pi}{14} = -7$	**M1 A1**
	But $\cot\dfrac{\pi}{14} = -\cot\dfrac{13\pi}{14}$, $\cot\dfrac{3\pi}{14} = -\cot\dfrac{11\pi}{14}$, $\cot\dfrac{5\pi}{14} = -\cot\dfrac{9\pi}{14}$	**M1**
	Hence $\cot^2\dfrac{1}{14}\pi\ \cot^2\dfrac{3}{14}\pi\cot^2\dfrac{5}{14}\pi = 7$. (**AG**)(Penultimate line must be seen.)	**A1**
	SC Award **B1** if product of roots mentioned, without proper pairing seen.	[5]

Q20.

9	$\left(z+\frac{1}{z}\right)^4 = \left(z^4+\frac{1}{z^4}\right)+4\left(z^2+\frac{1}{z^2}\right)+6$	**M1**
	Taking $z=\cos\theta+\mathrm{i}\sin\theta$, $(2\cos\theta)^4 = 2\cos 4\theta+8\cos 2\theta+6$	**M1A1**
	$\Rightarrow \cos^4\theta = \frac{1}{16}(2\cos 4\theta+8\cos 2\theta+6) = \frac{1}{8}(\cos 4\theta+4\cos 2\theta+3)$ (**AG**)	**A1** [4]
	$\left(z-\frac{1}{z}\right)^4 = \left(z^4+\frac{1}{z^4}\right)-4\left(z^2+\frac{1}{z^2}\right)+6$	**M1**
	$\Rightarrow (2\mathrm{i}\sin\theta)^4 = 2\cos 4\theta-8\cos 2\theta+6$	**M1A1**
	$\Rightarrow \sin^4\theta = \frac{1}{16}(2\cos 4\theta-8\cos 2\theta+6) = \frac{1}{8}(\cos 4\theta-4\cos 2\theta+3)$	**A1** [4]
	$\int_0^{\frac{1}{8}\pi}\left(\cos^4\theta+\sin^4\theta\right)\,\mathrm{d}\theta = \frac{1}{4}\int_0^{\frac{1}{8}\pi}(\cos 4\theta+3)\,\mathrm{d}\theta$	**M1**
	$= \frac{1}{4}\left[\frac{\sin 4\theta}{4}+3\theta\right]_0^{\frac{1}{8}\pi} = \frac{1}{16}+\frac{3}{32}\pi$ (oe)	**M1A1** [3]

Q21.

10	$z^n = \cos n\theta + \mathrm{i}\sin n\theta$ and $z^{-n} = \cos(-n\theta) + \mathrm{i}\sin(-n\theta) = \cos n\theta - \mathrm{i}\sin n\theta$ $\Rightarrow\ z^n + \frac{1}{z^n} = 2\cos n\theta$ and $z^n - \frac{1}{z^n} = 2\mathrm{i}\sin n\theta$. (AG)	**M1A1**	[2]
	$\left(z-\frac{1}{z}\right)^4\left(z+\frac{1}{z}\right)^2 = \left(z-\frac{1}{z}\right)^2\left(z^2-\frac{1}{z^2}\right)^2$ (Or 1st bracket expanded) $= \left(z^2-2+\frac{1}{z^2}\right)\left(z^4-2+\frac{1}{z^4}\right)$ (and 2nd bracket expanded) $= \left(z^6+\frac{1}{z^6}\right)-2\left(z^4+\frac{1}{z^4}\right)-\left(z^2+\frac{1}{z^2}\right)+4$ (Expanded and grouped) $\Rightarrow\ 64\sin^4\theta\cos^2\theta = 2\cos 6\theta - 4\cos 4\theta - 2\cos 2\theta + 4$ $\Rightarrow\ \sin^4\theta\cos^2\theta = \frac{1}{32}(\cos 6\theta - 2\cos 4\theta - \cos 2\theta + 2)$ (AG) (Uses initial result)	**M1** **M1** **M1A1** **M1A1** **A1**	[7]
	$\int_0^{\frac{1}{4}\pi} \sin^4\theta\cos^2\theta\,\mathrm{d}\theta = \frac{1}{32}\left[\frac{1}{6}\sin 6\theta - \frac{1}{2}\sin 4\theta - \frac{1}{2}\sin 2\theta + 2\theta\right]_0^{\frac{1}{4}\pi}$ $= \frac{1}{32}\left(-\frac{1}{6} - 0 - \frac{1}{2} + \frac{\pi}{2}\right) = \left(\frac{3\pi-4}{192}\right)$ or 0.0283 (3sf) ALT METHOD: Expand and group each bracket separately and sub trig: **M1, A1, A1** Multiply together two brackets and manipulate trig expressions **M1,M1** LHS = RHS **M1 A1**	**M1A1** **A1**	[3]

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Q22.

8	$\sum_{r=1}^{n} z^{2r-1} = \frac{z\left(1-[z^2]^n\right)}{1-z^2} = \frac{z-z^{2n+1}}{1-z^2}$	M1A1
	$= \frac{1-z^{2n}}{z^{-1}-z} = \frac{1-(\cos 2n\theta + i\sin 2n\theta)}{\cos\theta - i\sin\theta - (\cos\theta + i\sin\theta)} = \frac{1-\cos 2n\theta - i\sin 2n\theta}{-2i\sin\theta}$	M1A1 A1
	Equating imaginary parts: $\sum_{r=1}^{n} \sin(2r-1)\theta = \frac{2\sin^2 n\theta}{2\sin\theta} = \frac{\sin^2 n\theta}{\sin\theta}$ (AG)	M1A1 (7)
	Differentiating: $\sum_{r=1}^{n} \sin(2r-1)\cos(2r-1)\theta = 2n\sin n\theta\cos n\theta\,\mathrm{cosec}\,\theta - \sin^2 n\theta\,\mathrm{cosec}\,\theta\cot\theta$	M1A1
	Putting $\theta = \frac{\pi}{2n}$: $\sum_{r=1}^{n} (2r-1)\cos(2r-1)\left(\frac{\pi}{2n}\right) = n\sin\frac{\pi}{2}\cos\frac{\pi}{2}\mathrm{cosec}\left(\frac{\pi}{2n}\right) - \sin^2\frac{\pi}{2}\mathrm{cosec}\left(\frac{\pi}{2n}\right)\cot\left(\frac{\pi}{2n}\right)$	dM1
	$\Rightarrow \sum_{r=1}^{n} (2r-1)\cos\left[\frac{(2r-1)\pi}{2n}\right] = -\mathrm{cosec}\left(\frac{\pi}{2n}\right)\cot\left(\frac{\pi}{2n}\right)$. (AG)	A1 (4)
		Total: 11

Q23.

6	$(1+z)^n = 1+\binom{n}{1}z+\binom{n}{2}z^2+\cdots+\binom{n}{n}z^n$	B1
	$\text{Re}\{(1+z)^n\} = 1+\binom{n}{1}\cos\theta+\binom{n}{2}\cos 2\theta+\cdots+\binom{n}{n}\cos n\theta$	M1A1
	$\text{Re}([1+\cos\theta]+\text{i}\sin\theta)^n = \text{Re}\left(2\cos^2\left[\tfrac{1}{2}\theta\right]+2\text{i}\sin\left[\tfrac{1}{2}\theta\right]\cos\left[\tfrac{1}{2}\theta\right]\right)^n$	M1A1
	$= 2^n\cos^n\left(\tfrac{1}{2}\theta\right)\text{Re}\left(\cos\left[\tfrac{1}{2}\theta\right]+\text{i}\sin\left[\tfrac{1}{2}\theta\right]\right)^n$	A1
	$\Rightarrow \binom{n}{1}\cos\theta+\binom{n}{2}\cos 2\theta+\cdots+\binom{n}{n}\cos n\theta = 2^n\cos^n\left(\tfrac{1}{2}\theta\right)\cos\left(\tfrac{1}{2}n\theta\right)-1$ (AG)	A1 (7)
	$\text{Im}\{(1+z)^n\} = \binom{n}{1}\sin\theta+\binom{n}{2}\sin 2\theta+\cdots+\binom{n}{n}\sin n\theta$	M1
	$\Rightarrow \binom{n}{1}\sin\theta+\binom{n}{2}\sin 2\theta+\cdots+\binom{n}{n}\sin n\theta = 2^n\cos^n\left(\tfrac{1}{2}\theta\right)\sin\left(\tfrac{1}{2}n\theta\right)$	A1 (2) **Total 9**

Q24.

Qn & Part	Solution	Marks
10	$(\cos\theta + \mathrm{i}\sin\theta)^5 = \cos 5\theta + \mathrm{i}\sin 5\theta$	**B1**
	$(c + \mathrm{i}s)^5 = c^5 + 5c^4 s\mathrm{i} - 10c^3 s^2 - 10\mathrm{i}c^2 s^3 \mathrm{i} + 5\mathrm{c}s^4 + s^5\mathrm{i}$	**M1A1**
	$\tan 5\theta = \dfrac{5c^4 s - 10c^2 s^3 + s^5}{c^5 - 10c^3 s^2 + 5cs^4}$	**M1**
	Divide numerator and denominator by c^5 (stated or shown):	
	$\Rightarrow \tan 5\theta = \dfrac{5\tan\theta - 10\tan^3\theta + \tan^5\theta}{1 - 10\tan^2\theta + 5\tan^4\theta}$ (AG)	**A1** [5]
	$\tan 5\theta = 0 \Rightarrow \theta = \dfrac{1}{5}\pi, \dfrac{2}{5}\pi, \dfrac{3}{5}\pi, \dfrac{4}{5}\pi, \pi$	**B1**
	$t^5 - 10t^3 + 5t = 0$ has roots $\tan\left(\dfrac{1}{5}\pi\right), \tan\left(\dfrac{2}{5}\pi\right), \tan\left(\dfrac{3}{5}\pi\right), \tan\left(\dfrac{4}{5}\pi\right), \tan\pi$	
	$\Rightarrow t^4 - 10t^2 + 5 = 0$ has roots $\tan\left(\dfrac{1}{5}\pi\right), \tan\left(\dfrac{2}{5}\pi\right), \tan\left(\dfrac{3}{5}\pi\right), \tan\left(\dfrac{4}{5}\pi\right)$.	**B1**
	$\Rightarrow \left(t^2 - \tan^2\left(\dfrac{1}{5}\pi\right)\right)\left(t^2 - \tan^2\left(\dfrac{2}{5}\pi\right)\right) = 0$	
	since $\tan\left(\dfrac{1}{5}\pi\right) = -\tan\left(\dfrac{4}{5}\pi\right)$ and $\tan\left(\dfrac{2}{5}\pi\right) = -\tan\left(\dfrac{3}{5}\pi\right)$.	**M1**
	$\Rightarrow x^2 - 10x + 5 = 0$ has roots $\tan^2\left(\dfrac{1}{5}\pi\right)$ and $\tan^2\left(\dfrac{2}{5}\pi\right)$. (AG)	**A1** [4]
	$\sec^2\alpha = 1 + \tan^2\alpha$	**M1**
	$y = 1 + x \Rightarrow x = y - 1 \Rightarrow (y-1)^2 - 10(y-1) + 5 = 0$	**M1**
	$\Rightarrow y^2 - 12y + 16 = 0$	**A1** [3] **Total 12**

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Q25.

7	$(\cos\theta + \text{i}\sin\theta)^5 = \cos 5\theta + \text{i}\sin 5\theta$ $= c^5 + 5c^4(\text{i}s) + 10c^3(\text{i}s)^4 + 10c^2(\text{i}s)^3 + 5c(\text{i}s)^4 + (\text{i}s)^5$	M1 A1
	$\tan 5\theta = \dfrac{\sin 5\theta}{\cos 5\theta} = \dfrac{5c^4s - 10c^2s^3 + s^5}{c^5 - 10c^3s^2 + 5cs^4} = \ldots = \dfrac{5t - 10t^3 + t^5}{1 - 10t^2 + 5t^4}$ (AG)	M1A1 [4]
	$\tan 5\theta = 0 \Rightarrow t(t^4 - 10t^2 + 5) = 0$ (Allow non-factorised form.) Rejecting $t = 0$, which implies $\theta = k\pi$, roots of $t^4 - 10t^2 + 5 = 0$ are: $\pm\tan \frac{1}{5}\pi$ and $\pm\tan \frac{2}{5}\pi$, since $5\theta = \pm 2\pi$ or $5\theta = \pm 4\pi$ (AG)	M1 A1 A1 [3]
	Product of roots $= \tan^2 \frac{1}{5}\pi \tan^2 \frac{2}{5}\pi = 5 \Rightarrow \tan \frac{1}{5}\pi \tan \frac{2}{5}\pi = \sqrt{5}$ (S.C. Award B1√ if result deduced from information given in the question.)	M1A1 [2]

Q26.

5	$\dfrac{z(z^n - 1)}{z - 1}$ (OE)	B1 [1]
	$\cos\theta + \cos 2\theta + \cos 3\theta + \ldots + \cos n\theta = \text{Re}\left\{\dfrac{z\left(z^n - 1\right)}{\left(z - 1\right)}\right\}$	M1
	$= \text{Re}\left\{\dfrac{z^{\frac{1}{2}}\left(z^n - 1\right)}{\left(z^{\frac{1}{2}} - z^{-\frac{1}{2}}\right)}\right\}$	M1
	$= \text{Re}\left\{\dfrac{z^{n+\frac{1}{2}} - z^{\frac{1}{2}}}{2\text{i}\sin\frac{1}{2}\theta}\right\}$	A1
	$= \dfrac{\sin\left(n + \frac{1}{2}\right)\theta - \sin\frac{1}{2}\theta}{2\sin\frac{1}{2}\theta}$	A1
	$= \dfrac{2\cos\frac{1}{2}(n+1)\theta\sin\frac{1}{2}n\theta}{2\sin\frac{1}{2}\theta}$ (Both previous M marks req.)	M1A1
	$= \dfrac{\cos\frac{1}{2}(n+1)\theta\sin\frac{1}{2}n\theta}{\sin\frac{1}{2}\theta}$ (AG)	A1 [7]

5	**Alternative (i)**	
	$\Sigma = \text{Re}\left\{\dfrac{z - z^{n+1}}{1-z}\right\}$	(M1)
	$= \text{Re}\left\{\dfrac{e^{i\theta} - e^{i(n+1)\theta}}{1-e^{i\theta}} \times \dfrac{1-e^{-i\theta}}{1-e^{-i\theta}}\right\}$	(M1)
	$= \text{Re}\left\{\dfrac{e^{i\theta} - e^{i(n+1)\theta} - 1 + e^{in\theta}}{2-2\cos\theta}\right\}$	
	$= \dfrac{\cos\theta - \cos(n+1)\theta - 1 + \cos n\theta}{2-2\cos\theta}$ (Numerator and denominator.)	(A1A1)
	$= \dfrac{-2\sin^2\left(\dfrac{\theta}{2}\right) + 2\sin\left(\dfrac{2n+1}{2}\right)\theta \sin\left(\dfrac{\theta}{2}\right)}{4\sin^2\left(\dfrac{\theta}{2}\right)}$	(A1)
	$= \dfrac{1}{2\sin\left(\dfrac{\theta}{2}\right)}\left\{\sin\left(n+\dfrac{1}{2}\right)\theta - \sin\left(\dfrac{\theta}{2}\right)\right\}$	(M1)
	$= \dfrac{2\cos\left(\dfrac{n+1}{2}\right)\theta \sin\left(\dfrac{n\theta}{2}\right)}{2\sin\left(\dfrac{\theta}{2}\right)}$	
	$= \dfrac{\cos\left(\dfrac{n+1}{2}\right)\theta \sin\left(\dfrac{n\theta}{2}\right)}{\sin\left(\dfrac{\theta}{2}\right)}$ (CAO) (AG)	(A1)
		[7]

Q27.

6	$(c+\mathrm{i}s)^5 = c^5 + 5c^4\mathrm{i}s + 10c^3(\mathrm{i}s)^2 + 10c^2(\mathrm{i}s)^3 + 5(\mathrm{i}s)^4 + (\mathrm{i}s)^5$	B1
	$\cos 5\theta = c^5 - 10c^3s^2 + 5cs^4$	M1A1
	$= c\left((1-s^2)^2 - 10s^2(1-s^2) + 5s^4\right)$	M1
	$= c\left(1 - 2s^2 + s^4 - 10s^2 + 10s^4 + 5s^4\right) = c\left(16s^4 - 12s^2 + 1\right)$ (AG)	A1
		(5)
	Alternative: using $\left(z + z^{-1}\right)^n = 2\cos n\theta$	
	$(2\cos\theta)^5 = 2\cos 5\theta + 10\cos 3\theta + 20\cos\theta$	(M1)
	$\Rightarrow 16\cos^5\theta = \cos 5\theta + 5\cos 3\theta + 10\cos\theta$	
	But $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$ (Can be quoted, if known.)	(B1)
	$\Rightarrow \cos 5\theta = \cos\theta\left(16\cos^4\theta - 20\cos^2\theta + 5\right)$	(A1)
	Uses $\cos^2\theta = 1 - \sin^2\theta$ to obtain $\cos 5\theta = c\left(16s^4 - 12s^2 + 1\right)$ (AG)	(M1A1)
	$\cos 5\theta = 0 \Rightarrow \theta = \frac{1}{10}\pi, \frac{3}{10}\pi, \frac{1}{2}\pi, \frac{7}{10}\pi, \frac{9}{10}\pi$	B1
	$s^2 = \dfrac{12 \pm \sqrt{144-64}}{32} = \dfrac{3 \pm \sqrt{5}}{8}$	M1A1
	Since $\sin^2\left(\frac{1}{10}\pi\right) < \sin^2\left(\frac{3}{10}\pi\right)$, $\sin^2\left(\frac{1}{10}\pi\right) = \frac{3-\sqrt{5}}{8}$ (Or by any other valid method.)	A1
	(AG) Justification required for final mark.	(4)
		[9]

DIFFERENTIAL EQUATIONS

3 The variables t and x are related by the differential equation

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \frac{\mathrm{d}x}{\mathrm{d}t} + x = t^2 + 1.$$

(a) Find the general solution for x in terms of t. [6]

(b) Deduce an approximate value of $\frac{\mathrm{d}^2x}{\mathrm{d}t^2}$ for large positive values of t. [2]

Q2.

6 Use the substitution $y = vx$ to find the solution of the differential equation

$$x\frac{\mathrm{d}y}{\mathrm{d}x} = y + \sqrt{9x^2 + y^2}$$

for which $y = 0$ when $x = 1$. Give your answer in the form $y = \mathrm{f}(x)$, where $\mathrm{f}(x)$ is a polynomial in x. [10]

Q3.

5 Find the solution of the differential equation

$$x(x+7)\frac{\mathrm{d}y}{\mathrm{d}x}+7y=x$$

for which $y=7$ when $x=1$. Give your answer in the form $y=\mathrm{f}(x)$. [9]

2 The variables x and y are related by the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 3\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = 2x + 1.$$

(a) Find the general solution for y in terms of x. [6]

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(b) State an approximate solution for large positive values of x. [1]

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Q5.

4 Find the solution of the differential equation

$$\sin\theta\frac{\mathrm{d}y}{\mathrm{d}\theta}+y=\tan\tfrac{1}{2}\theta,$$

where $0<\theta<\pi$, given that $y=1$ when $\theta=\frac{1}{2}\pi$. Give your answer in the form $y=\mathrm{f}(\theta)$. [9]

[You may use without proof the result that $\int\operatorname{cosec}\theta\,\mathrm{d}\theta=\ln\tan\tfrac{1}{2}\theta$.]

Q6.

7 The variables x and y are related by the differential equation

$$4\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - y = 3.$$

It is given that, when $x = 0$, $y = -3$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = 2$.

(a) Find y in terms of x. [8]

(b) Deduce the exact value of x for which $y = 0$. Give your answer in logarithmic form. [3]

Q7.

5 The variables x and y are related by the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 2\frac{\mathrm{d}y}{\mathrm{d}x} - 3y = 4\mathrm{e}^{-x}.$$

(a) Find the value of the constant k such that $y = kx\mathrm{e}^{-x}$ is a particular integral of the differential equation. [4]

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(b) Find the solution of the differential equation for which $y = \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2}$ when $x = 0$. [6]

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5 Find the particular solution of the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 2\frac{\mathrm{d}y}{\mathrm{d}x} + y = 4\cos x,$$

given that, when $x = 0$, $y = -4$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = 3$. [11]

Q9.

7 **(a)** Show that an appropriate integrating factor for

$$\sqrt{x^2-1}\,\frac{\mathrm{d}y}{\mathrm{d}x}+y=x^2-x\sqrt{x^2-1}$$

is $x+\sqrt{x^2-1}$. [4]

(b) Hence find the solution of the differential equation

$$\sqrt{x^2-1}\,\frac{\mathrm{d}y}{\mathrm{d}x}+y=x^2-x\sqrt{x^2-1}$$

for which $y=1$ when $x=\frac{5}{4}$. Give your answer in the form $y=\mathrm{f}(x)$. [7]

Q10.

2 Find the solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{4x^3 y}{x^4 + 5} = 6x$$

for which $y = 1$ when $x = 1$. Give your answer in the form $y = \mathrm{f}(x)$. [7]

(b) Find the general solution for w in terms of x. [7]

Q11.

7 It is given that $y = x^2 w$ and

$$x^2 \frac{\mathrm{d}^2 w}{\mathrm{d}x^2} + 4x(x+1)\frac{\mathrm{d}w}{\mathrm{d}x} + (5x^2 + 8x + 2)w = 5x^2 + 4x + 2.$$

(a) Show that

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 4\frac{\mathrm{d}y}{\mathrm{d}x} + 5y = 5x^2 + 4x + 2. \qquad [4]$$

Q12.

1 Find the solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + 5y = \mathrm{e}^{-7x}$$

for which $y = 0$ when $x = 0$. Give your answer in the form $y = \mathrm{f}(x)$. [6]

Q13.

7 It is given that $x = t^3 y$ and

$$t^3 \frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + \left(4t^3 + 6t^2\right)\frac{\mathrm{d}y}{\mathrm{d}t} + \left(13t^3 + 12t^2 + 6t\right)y = 61\mathrm{e}^{\frac{1}{2}t}.$$

(a) Show that

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + 4\frac{\mathrm{d}x}{\mathrm{d}t} + 13x = 61\mathrm{e}^{\frac{1}{2}t}.$$ [4]

(b) Find the general solution for y in terms of t. [7]

Q14.

1 Find the general solution of the differential equation

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} - 8\frac{\mathrm{d}x}{\mathrm{d}t} - 9x = 9\mathrm{e}^{8t}.$$ [6]

Q15.

7 **(a)** Show that an appropriate integrating factor for

$$(x^2+1)\frac{\mathrm{d}y}{\mathrm{d}x}+y\sqrt{x^2+1}=x^2-x\sqrt{x^2+1}$$

is $x+\sqrt{x^2+1}$. [4]

(b) Hence find the solution of the differential equation

$$(x^2+1)\frac{\mathrm{d}y}{\mathrm{d}x}+y\sqrt{x^2+1}=x^2-x\sqrt{x^2+1}$$

for which $y=\ln 2$ when $x=0$. Give your answer in the form $y=\mathrm{f}(x)$. [7]

Q16.

7 Find the particular solution of the differential equation

$$10\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 3\frac{\mathrm{d}x}{\mathrm{d}t} - x = t + 2,$$

given that when $t = 0$, $x = 0$ and $\frac{\mathrm{d}x}{\mathrm{d}t} = 0$. [10]

Q17.

8 Find the particular solution of the differential equation

$$9\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 6\frac{\mathrm{d}x}{\mathrm{d}t} + x = 50\sin t,$$

given that when $t = 0$, $x = 0$ and $\frac{\mathrm{d}x}{\mathrm{d}t} = 0$. [10]

Q18.

It is given that $w = \cos y$ and

$$\tan y \frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + 2\tan y \frac{\mathrm{d}y}{\mathrm{d}x} = 1 + \mathrm{e}^{-2x}\sec y.$$

(i) Show that

$$\frac{\mathrm{d}^2w}{\mathrm{d}x^2} + 2\frac{\mathrm{d}w}{\mathrm{d}x} + w = -\mathrm{e}^{-2x}. \quad [4]$$

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(ii) Find the particular solution for y in terms of x, given that when $x = 0$, $y = \frac{1}{3}\pi$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\sqrt{3}}$. [10]

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Q19.

7 Find the particular solution of the differential equation

$$49\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 14\frac{\mathrm{d}y}{\mathrm{d}x} + y = 49x + 735,$$

given that when $x = 0$, $y = 0$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$. [10]

Q20.

10 It is given that $t \neq 0$ and

$$t\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 2\frac{\mathrm{d}x}{\mathrm{d}t} + 9tx = 3t^2 + 1.$$

(i) Show that if $y = tx$ then

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + 9y = 3t^2 + 1. \qquad [3]$$

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(ii) Find x in terms of t, given that $x = \frac{1}{9}\pi$ and $\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{2}{3}$ when $t = \frac{1}{3}\pi$. [9]

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Q21.

10 **(i)** Find the particular solution of the differential equation

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 2\frac{\mathrm{d}x}{\mathrm{d}t} + 10x = 37\sin 3t,$$

given that $x = 3$ and $\frac{\mathrm{d}x}{\mathrm{d}t} = 0$ when $t = 0$. [10]

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(ii) Show that, for large positive values of t and for any initial conditions,

$$x \approx \sqrt{(37)}\sin(3t - \phi),$$

where the constant ϕ is such that $\tan\phi = 6$. [3]

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Q22.

4 **(i)** Find the general solution of the differential equation

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 2\frac{\mathrm{d}x}{\mathrm{d}t} + x = 4\sin t. \qquad [7]$$

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(ii) State an approximate solution for large positive values of t. [1]

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Q23.

10 It is given that $x = t^{\frac{1}{2}}$, where $x > 0$ and $t > 0$, and y is a function of x.

(i) Show that $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 2t^{\frac{1}{2}}\dfrac{\mathrm{d}y}{\mathrm{d}t}$ and $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = 2\dfrac{\mathrm{d}y}{\mathrm{d}t} + 4t\dfrac{\mathrm{d}^2y}{\mathrm{d}t^2}$. [3]

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(ii) Hence show that the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - \left(8x + \frac{1}{x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} + 12x^2y = 4x^2\mathrm{e}^{-x^2} \qquad (*)$$

reduces to the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} - 4\frac{\mathrm{d}y}{\mathrm{d}t} + 3y = \mathrm{e}^{-t}.$$ [1]

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(iii) Find the general solution of $(*)$, giving y in terms of x. [7]

Q24.

8 Find the solution of the differential equation

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 6\frac{\mathrm{d}x}{\mathrm{d}t} + 9x = 18t^2 + 6t + 1,$$

given that, when $t = 0$, $x = 3$ and $\frac{\mathrm{d}x}{\mathrm{d}t} = 0$. [10]

Q25.

2 Find the general solution of the differential equation

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 2\frac{\mathrm{d}x}{\mathrm{d}t} + 5x = 4 - 5t^2. \qquad [6]$$

3 **(i)** Show that $\dfrac{\mathrm{d}^{n+1}}{\mathrm{d}x^{n+1}}(x^{n+1}\ln x) = \dfrac{\mathrm{d}^n}{\mathrm{d}x^n}\left(x^n + (n+1)x^n \ln x\right)$. [2]

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(ii) Prove by mathematical induction that, for all positive integers n,

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}(x^n \ln x) = n!\left(\ln x + 1 + \frac{1}{2} + \ldots + \frac{1}{n}\right).$$ [5]

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Q26.

9 Find the value of the constant k such that $y = kx^2e^{2x}$ is a particular integral of the differential equation

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 4e^{2x}. \qquad (*)$$ [4]

Hence find the general solution of $(*)$. [3]

Find the particular solution of $(*)$ such that $y = 3$ and $\frac{dy}{dx} = -2$ when $x = 0$. [4]

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Q27.

10 Given that y is a function of x and that $x = \mathrm{e}^u$, show that

$$x\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \qquad \text{and} \qquad x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\mathrm{d}^2y}{\mathrm{d}u^2} - \frac{\mathrm{d}y}{\mathrm{d}u}. \qquad [3]$$

Given also that

$$x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 3x\frac{\mathrm{d}y}{\mathrm{d}x} + 17y = 34\ln x + 21,$$

deduce that

$$\frac{\mathrm{d}^2y}{\mathrm{d}u^2} + 2\frac{\mathrm{d}y}{\mathrm{d}u} + 17y = 34u + 21. \qquad [1]$$

Find y in terms of x given that $y = 0$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = -1$ when $x = 1$. [9]

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Q28.

6 Find the general solution of the differential equation

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 7\frac{\mathrm{d}x}{\mathrm{d}t} + 10x = 116\sin 2t. \qquad [8]$$

State an approximate solution for large positive values of t. [1]

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Q29.

11 Show that the substitution $v = \dfrac{1}{y}$ reduces the differential equation

$$\frac{2}{y^3}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 - \frac{1}{y^2}\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - \frac{2}{y^2}\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{5}{y} = 17 + 6x - 5x^2$$

to the differential equation

$$\frac{\mathrm{d}^2v}{\mathrm{d}x^2} + 2\frac{\mathrm{d}v}{\mathrm{d}x} + 5v = 17 + 6x - 5x^2. \qquad [4]$$

Hence find y in terms of x, given that when $x = 0$, $y = \frac{1}{2}$ and $\dfrac{\mathrm{d}y}{\mathrm{d}x} = -1$. [10]

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Q30.

9 Find the particular solution of the differential equation

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} - 3\frac{\mathrm{d}x}{\mathrm{d}t} - 10x = 2\sin t - 3\cos t,$$

given that, when $t = 0$, $x = 3.3$ and $\frac{\mathrm{d}x}{\mathrm{d}t} = 0.9$. [11]

Q31.

2 Find the general solution of the differential equation

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 4\frac{\mathrm{d}x}{\mathrm{d}t} + 4x = 7 - 2t^2. \qquad [6]$$

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Q32.

4 Obtain the general solution of the differential equation

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} - 6\frac{\mathrm{d}x}{\mathrm{d}t} + 25x = 195\sin 2t. \qquad [6]$$

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Q33.

10 Find the particular solution of the differential equation

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 0.16\frac{\mathrm{d}x}{\mathrm{d}t} + 0.0064x = 8.64 + 0.32t,$$

given that when $t = 0$, $x = 0$ and $\frac{\mathrm{d}x}{\mathrm{d}t} = 0$. [10]

Show that, for large positive t, $\frac{\mathrm{d}x}{\mathrm{d}t} \approx 50$. [2]

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Q34.

9 Given that

$$x\frac{d^2y}{dx^2} + (2x + 2)\frac{dy}{dx} + (2 - 3x)y = 10e^{2x}$$

and that $v = xy$, show that

$$\frac{d^2v}{dx^2} + 2\frac{dv}{dx} - 3v = 10e^{2x}. \quad [4]$$

Find the general solution for y in terms of x. [7]

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DIFFERENTIAL EQUATIONS
MS

Q1.

PUBLISHED

Question	Answer	Marks	Guidance
3(a)	$z^2+z+1=0$ leading to $m=-\frac{1}{2}\pm i\frac{\sqrt{3}}{2}$	M1	Auxiliary equation.
	$x=e^{-\frac{1}{2}t}\left(A\sin\frac{\sqrt{3}}{2}t+B\cos\frac{\sqrt{3}}{2}t\right)$	A1	Complimentary function.
	$x=pt^2+qt+r$ leading to $\dot{x}=2pt+q$ leading to $\ddot{x}=2p$	B1	Particular integral and its derivatives.
	$2p+2pt+q+pt^2+qt+r=t^2+1$	M1	Substitutes and equates coefficients. PI must have the correct form.
	$r=1,\quad q=-2,\quad p=1$	A1	
	$x=e^{-\frac{1}{2}t}\left(A\sin\frac{\sqrt{3}}{2}t+B\cos\frac{\sqrt{3}}{2}t\right)+t^2-2t+1$	A1	
		6	
3(b)	$e^{-\frac{1}{2}t}\to 0$ as $t\to\infty$	M1	
	$\ddot{x}\to 2$	A1 FT	
		2	

Q2.

Question	Answer	Marks	Guidance
6	$\frac{dy}{dx}=v+x\frac{dv}{dx}$	B1	
	$x\left(v+x\frac{dv}{dx}\right)=vx+\sqrt{9x^2+x^2v^2}$	M1	Substitutes and derives first order separable equation.
	$x\frac{dv}{dx}=\sqrt{v^2+9}$	A1	
	$\int\frac{1}{\sqrt{v^2+9}}dv=\int\frac{1}{x}dx$	M1	Separates variables and integrates both sides.
	$\sinh^{-1}\left(\frac{v}{3}\right)=\ln x+C$	A1	
	$C=0$	B1	Substitutes initial conditions.
	$\ln\left(\frac{y}{3}+\sqrt{\frac{y^2}{9}+1}\right)=\ln x$	M1	Converts to logarithmic form.
	$y+\sqrt{y^2+9x^2}=3x^2$	A1	
	$9x^4-6x^2y+y^2-y^2-9x^2=0$	M1	Squares to eliminate radical.
	$y=\frac{3}{2}(x^2-1)$	A1	
		10	

Q3.

Question	Answer	Marks	Guidance
5	$\frac{dy}{dx}+\frac{7y}{x(x+7)}=\frac{1}{x+7}$	**B1**	Divides through by $x(x+7)$.
	$\frac{7}{x(x+7)}=\frac{1}{x}-\frac{1}{x+7}$ or $\frac{7}{x(x+7)}=\frac{7}{\left(x+\frac{7}{2}\right)^2-\frac{49}{4}}$	**B1**	Converts to partial fractions or completes the square.
	$e^{\int 7x^{-1}(x+7)^{-1}dx}=e^{\ln x-\ln(x+7)}=\frac{x}{x+7}$	**M1 A1**	Finds integrating factor. Accept non-zero multiple of $\frac{x}{x+7}$.
	$\frac{d}{dx}\left(\frac{yx}{x+7}\right)=\frac{x}{(x+7)^2}=\frac{1}{x+7}-\frac{7}{(x+7)^2}$	**M1**	Correct form on LHS and attempt to integrate RHS. RHS must be of the form $\frac{a}{x+7}+\frac{b}{(x+7)^2}$.
	$\frac{yx}{x+7}=\ln(x+7)+7(x+7)^{-1}\quad(+C)$	**A1**	
	$\frac{7}{8}=\ln 8+\frac{7}{8}+C$	**M1**	Finds C, substitutes into their expression.
	$y=\left(1+\frac{7}{x}\right)\ln\left(\frac{x+7}{8}\right)+\frac{7}{x}$	**M1 A1**	Divides their expression through by coefficient of y.
		9	

Q4.

Question	Answer	Marks	Guidance
2(a)	$m^2+3m+2=0$ leading to m = -1, -2	**M1**	Auxiliary equation.
	$y=Ae^{-x}+Be^{-2x}$	**A1**	Complimentary function.
	$y=p+qx$ leading to $y'=q$ leading to $y''=0$	**B1**	Particular integral and its derivatives.
	$3q+2(p+qx)=2x+1$	**M1**	Substitutes and equates coefficients.
	$q=1 \quad p=-1$	**A1**	
	$y=Ae^{-x}+Be^{-2x}+x-1$	**A1**	Must see '$y=$'.
		6	

Question	Answer	Marks	Guidance
2(b)	$y=x-1$	**B1 FT**	Accept $y \approx x-1$.
		1	

Question	Answer	Marks	Guidance
4	$\frac{\mathrm{d}y}{\mathrm{d}\theta} + \operatorname{cosec}\theta y = \frac{\tan\frac{1}{2}\theta}{\sin\theta}$	**B1**	Divides through by $\sin\theta$.
	$\mathrm{e}^{\int \operatorname{cosec}\theta \mathrm{d}\theta} = \mathrm{e}^{\ln\tan\frac{1}{2}\theta} = \tan\frac{1}{2}\theta$	**M1 A1**	Finds integrating factor.
	$\frac{\mathrm{d}}{\mathrm{d}\theta}\left(y\tan\frac{1}{2}\theta\right) = \frac{\tan^2\frac{1}{2}\theta}{\sin\theta} = \frac{1}{2}\tan\frac{1}{2}\theta\sec^2\frac{1}{2}\theta \quad \left[= \frac{\frac{1}{2}\sin\frac{1}{2}\theta}{\cos^3\frac{1}{2}\theta}\right]$	**M1**	Correct form on LHS and uses an appropriate identity.
	$y\tan\frac{1}{2}\theta = \frac{1}{2}\tan^2\frac{1}{2}\theta + C \qquad \left(\text{or} \quad \frac{1}{2}\sec^2\frac{1}{2}\theta + C\right)$	**M1 A1**	Integrates RHS.
	$1 = \frac{1}{2} + C$	**M1**	Substitutes initial conditions.
	$y = \frac{1}{2}\left(\tan\frac{1}{2}\theta + \cot\frac{1}{2}\theta\right)\left[= \operatorname{cosec}\theta\right]$	**M1 A1**	Divides through by their integrating factor.
		9	

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Q6.

Question	Answer	Marks	Guidance
7(a)	$4m^2 - 1 = 0$ leading to $m = \pm\frac{1}{2}$,	**M1**	Auxiliary equation.
	$y = Ae^{-\frac{1}{2}x} + Be^{\frac{1}{2}x}$	**A1**	Complementary function. Allow "$y =$" missing.
	$-k = 3$ leading to $k = -3$	**M1 A1**	Substitutes $y = k$.
	$y = Ae^{-\frac{1}{2}x} + Be^{\frac{1}{2}x} - 3$	**A1 FT**	General solution. Must have $y =$ or sensible equivalent, such as $y_{GS} =$. Don't allow $GS =$.
	$y' = -\frac{1}{2}Ae^{-\frac{1}{2}x} + \frac{1}{2}Be^{\frac{1}{2}x}$	**B1**	Derivative of general solution.
	$A + B = 0 \quad -\frac{1}{2}A + \frac{1}{2}B = 2$ leading to $A = -2, B = 2$	**M1**	Substitutes initial conditions into general solution, solves simultaneous equations.
	$y = 2e^{\frac{1}{2}x} - 2e^{-\frac{1}{2}x} - 3$	**A1**	Particular solution. Must have $y =$ or sensible equivalent, such as $y_{PS} =$. Don't allow $PS =$.
		8	
7(b)	$y = 0$ leading to $4\sinh\frac{1}{2}x = 3$ leading to $\frac{1}{2}x = \sinh^{-1}\frac{3}{4}$	**M1**	Makes x the subject. y must be of the form $a_1e^{bx} \pm a_2e^{-bx} + c$.
	$\sinh^{-1}\frac{3}{4} = \ln\left(\frac{3}{4} + \sqrt{\frac{9}{16} + 1}\right) = \ln 2$	**M1**	Uses logarithmic form. y must be of the form $a_1e^{bx} \pm a_2e^{-bx} + c$.
	$x = 2\ln 2 = \ln 4$	**A1**	
		3	

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Q7.

Question	Answer	Marks	Guidance
5(a)	$y = kx\mathrm{e}^{-x}$ leading to $y' = k\mathrm{e}^{-x}(1-x)$ leading to $y'' = k\mathrm{e}^{-x}(x-2)$	**B1 B1**	Differentiates particular integral.
	$k\mathrm{e}^{-x}(x-2) - 2k\mathrm{e}^{-x}(1-x) - 3kx\mathrm{e}^{-x} = 4\mathrm{e}^{-x}$ leading to $-4k = 4$	**M1**	Substitutes.
	$k = -1$	**A1**	
		4	
5(b)	$m^2 - 2m - 3 = 0$ leading to $m = -1, 3$	**M1**	Auxiliary equation.
	$y = A\mathrm{e}^{-x} + B\mathrm{e}^{3x}$	**A1**	Complementary function. (Can be awarded if seen in the general solution.)
	$y = A\mathrm{e}^{-x} + B\mathrm{e}^{3x} - x\mathrm{e}^{-x} = (A-x)\mathrm{e}^{-x} + B\mathrm{e}^{3x}$	**A1 FT**	General solution. FT on *their k*. Must have "$y =$".
	$y' = -(A-x)\mathrm{e}^{-x} - \mathrm{e}^{-x} + 3B\mathrm{e}^{3x}$	**B1**	Derivative of general solution.
	$A + B = \frac{1}{2}$, $-A - 1 + 3B = \frac{1}{2}$ leading to $A = 0,\ B = \frac{1}{2}$	**M1**	Substitutes initial conditions into general solution.
	$y = \frac{1}{2}\mathrm{e}^{3x} - x\mathrm{e}^{-x}$	**A1**	Must have "$y =$".
		6	

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Q8.

Question	Answer	Marks	Guidance
5	$m^2 - 2m + 1 = 0 \Rightarrow m = 1$	**M1**	Auxiliary equation.
	$y = \mathrm{e}^x (Ax + B)$	**A1**	Complementary function. Accept '$y =$' missing.
	$y = p\sin x + q\cos x$, $y' = p\cos x - q\sin x$, $y'' = -p\sin x - q\cos x$	**M1 A1**	Particular integral and its derivatives.
	$-p + 2q + p = 0 \quad -q - 2p + q = 4$	**M1**	Substitutes and equates coefficients.
	$p = -2 \quad q = 0$	**A1**	
	$y = \mathrm{e}^x (Ax + B) - 2\sin x$	**A1**	
	$y' = A\mathrm{e}^x + \mathrm{e}^x (Ax + B) - 2\cos x$	**M1**	Differentiates.
	$B = -4 \quad A + B - 2 = 3 \Rightarrow A = 9$	**M1 A1**	Forms simultaneous equations using initial conditions.
	$y = \mathrm{e}^x (9x - 4) - 2\sin x$	**A1**	
		11	

Question	Answer	Marks	Guidance
7(a)	$\frac{\mathrm{d}y}{\mathrm{d}x}+\frac{1}{\sqrt{x^2-1}}y=\frac{x^2-x\sqrt{x^2-1}}{\sqrt{x^2-1}}$	**B1**	Divides through by $\sqrt{x^2-1}$.
	$\mathrm{e}^{\int\frac{1}{\sqrt{(x^2-1)}}\mathrm{d}x}=\mathrm{e}^{\cosh^{-1}x}$	**M1 A1**	Finds integrating factor. M1 for correct form $\mathrm{e}^{a\int\frac{1}{\sqrt{(x^2-1)}}\mathrm{d}x}$, where a is a non-zero constant.
	$x+\sqrt{x^2-1}$	**A1**	AG
		4	
7(b)	$\frac{\mathrm{d}}{\mathrm{d}x}\left(y\left(x+\sqrt{x^2-1}\right)\right)=\frac{x}{\sqrt{x^2-1}}$	**M1 A1**	Correct form on LHS and simplifies RHS.
	$y\left(x+\sqrt{x^2-1}\right)=\sqrt{x^2-1}+C$	**M1 A1**	Integrates RHS. For M1, needs to integrate non-zero multiple of $\frac{x}{\sqrt{x^2-1}}$.
	$2=\frac{3}{4}+C$ leading to $C=\frac{5}{4}$	**M1**	Substitutes initial conditions.
	$y=\frac{\sqrt{x^2-1}+\frac{5}{4}}{x+\sqrt{x^2-1}}$	**M1 A1**	Divides through by coefficient of y.
		7	

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Q10.

Question	Answer	Marks	Guidance
2	$e^{\ln\left(x^4+5\right)} = x^4 + 5$	**M1 A1**	Finds integrating factor.
	$\frac{d}{dx}\left(y\left(x^4+5\right)\right) = 6x^5 + 30x$	**M1**	Writes in correct form.
	$y(x^4+5) = x^6 + 15x^2 + C$	**A1**	
	$6 = 16 + C$	**M1**	Substitutes initial conditions.
	$y = \frac{x^6 + 15x^2 - 10}{x^4 + 5}$	**M1 A1**	Division through by coefficient of y.
		7	

Question	Answer	Marks	Guidance
7(a)	$\frac{\mathrm{d}y}{\mathrm{d}x} = x^2\frac{\mathrm{d}w}{\mathrm{d}x} + 2xw$	**B1**	$\frac{\mathrm{d}w}{\mathrm{d}x} = \frac{1}{x^2}\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{2}{x^3}y$
	$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = x^2\frac{\mathrm{d}^2w}{\mathrm{d}x^2} + 4x\frac{\mathrm{d}w}{\mathrm{d}x} + 2w$	**B1**	$\frac{\mathrm{d}^2w}{\mathrm{d}x^2} = \frac{1}{x^2}\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - \frac{4}{x^3}\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{6}{x^4}y$
	$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 4\frac{\mathrm{d}y}{\mathrm{d}x} + 5y = x^2\frac{\mathrm{d}^2w}{\mathrm{d}x^2} + 4x\frac{\mathrm{d}w}{\mathrm{d}x} + 2w + 4x^2\frac{\mathrm{d}w}{\mathrm{d}x} + 8xw + 5x^2w$	**M1**	Uses substitution to find *w*-*x* equation, AG.
	$5x^2\frac{\mathrm{d}^2w}{\mathrm{d}x^2} + \left(4x^2+4x\right)\frac{\mathrm{d}w}{\mathrm{d}x} + \left(5x^2+8x+2\right)w = 5x^2+4x+2$	**A1**	AG
		4	
7(b)	$m^2+4m+5=0$ leading to $m=-2\pm i$	**M1**	Auxiliary equation.
	$y = \mathrm{e}^{-2x}\left(A\cos x + B\sin x\right)$	**A1**	Complimentary function. Allow "$y=$" missing. Don't allow e.g. $\mathrm{e}^{-2x}\left(A\cos x + Bi\sin x\right)$.
	$y = px^2+qx+r$, $y' = 2px+q$, $y'' = 2p$	**B1**	Particular integral and its derivatives.
	$2p+8px+4q+5px^2+5qx+5r = 5px^2+(8p+5q)x+(2p+4q+5r)$ $5p=5 \quad 8p+5q=4 \quad 2p+4q+5r=2$	**M1**	Substitutes and equates coefficients.
	$p=1 \qquad q=-\frac{4}{5} \qquad r=\frac{16}{25}$	**A1**	
	$x^2w = \mathrm{e}^{-2x}\left(A\cos x + B\sin x\right) + x^2 - \frac{4}{5}x + \frac{16}{25}$	**M1**	Substitutes for *y* and find *w* in terms of *x*.
	$w = \left(xe^x\right)^{-2}\left(A\cos x + B\sin x\right) + 1 - (\frac{5}{4}x)^{-1} + \left(\frac{5}{4}x\right)^{-2}$	**A1**	
		7	

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Q12.

Question	Answer	Marks
1	$e^{\int 5dx} = e^{5x}$	**M1 A1**
	$\frac{d}{dx}\left(ye^{5x}\right) = e^{-2x}$	**M1**
	$ye^{5x} = -\frac{1}{2}e^{-2x} + C$	**A1**
	$0 = -\frac{1}{2} + C$	**M1**
	$y = \frac{1}{2}e^{-5x} - \frac{1}{2}e^{-7x}$	**A1**
		6

Question	Answer	Marks
7(a)	$\frac{\mathrm{d}x}{\mathrm{d}t} = t^3\frac{\mathrm{d}y}{\mathrm{d}t} + 3t^2y$	**B1**
	$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = t^3\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + 6t^2\frac{\mathrm{d}y}{\mathrm{d}t} + 6ty$	**B1**
	$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 4\frac{\mathrm{d}x}{\mathrm{d}t} + 13x = t^3\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + 6t^2\frac{\mathrm{d}y}{\mathrm{d}t} + 6ty + 4t^3\frac{\mathrm{d}y}{\mathrm{d}t} + 12t^2y + 13t^3y = 61\mathrm{e}^{\frac{1}{2}t}$	**M1 A1**
		4
7(b)	$m^2 + 4m + 13 = 0 \Rightarrow m = -2 \pm 3i$	**M1**
	$x = \mathrm{e}^{-2t}\left(A\cos 3t + B\sin 3t\right)$	**A1**
	$x = k\mathrm{e}^{\frac{1}{2}t} \Rightarrow \dot{x} = \frac{1}{2}k\mathrm{e}^{\frac{1}{2}t} \Rightarrow \ddot{x} = \frac{1}{4}k\mathrm{e}^{\frac{1}{2}t}$	**B1**
	$\frac{1}{4}k + 2k + 13k = 61 \Rightarrow k = 4$	**M1 A1**
	$t^3y = \mathrm{e}^{-2t}\left(A\cos 3t + B\sin 3t\right) + 4\mathrm{e}^{\frac{1}{2}t} \Rightarrow y = t^{-3}\mathrm{e}^{-2t}\left(A\cos 3t + B\sin 3t\right) + 4t^{-3}\mathrm{e}^{\frac{1}{2}t}$	**M1 A1**
		7

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Q14.

Question	Answer	Marks
1	$m^2-8m-9=0\Rightarrow m=-1,9$	**M1**
	$x = Ae^{-t} + Be^{9t}$	**A1**
	$x = ke^{8t} \Rightarrow \dot{x} = 8ke^{8t} \Rightarrow \ddot{x} = 64ke^{8t}$	**B1**
	$64ke^{8t} - 64ke^{8t} - 9ke^{8t} = 9e^{8t} \Rightarrow -9k = 9$	**M1**
	$k = -1$	**A1**
	$x = Ae^{-t} + Be^{9t} - e^{8t}$	**A1**
		6

Question	Answer	Marks
7(a)	$\frac{\mathrm{d}y}{\mathrm{d}x}+\frac{y}{\sqrt{(x^2+1)}}=\frac{x}{x^2+1}\left(x-\sqrt{(x^2+1)}\right)$	**B1**
	$\mathrm{e}^{\int\frac{1}{\sqrt{(x^2+1)}}\mathrm{d}x}=\mathrm{e}^{\sinh^{-1}x}$	**M1 A1**
	$=x+\sqrt{(x^2+1)}$	**A1**
		4
7(b)	$\frac{\mathrm{d}}{\mathrm{d}x}\left(y\left(x+\sqrt{(x^2+1)}\right)\right)=-\frac{x}{x^2+1}$	**M1 A1**
	$y\left(x+\sqrt{(x^2+1)}\right)=-\int\frac{x}{x^2+1}\mathrm{d}x=-\frac{1}{2}\ln\left(x^2+1\right)+C$	**M1 A1**
	$\ln 2=C$	**M1**
	$y=\frac{\ln 2-\frac{1}{2}\ln\left(x^2+1\right)}{x+\sqrt{(x^2+1)}}=\left(x-\sqrt{(x^2+1)}\right)\ln\left(\frac{1}{2}\sqrt{(x^2+1)}\right)$	**M1 A1**
		7

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Question	Answer	Marks	Guidance
7	$10u^2+3u-1=0\Rightarrow(2u+1)(5u-1)=0$	**M1**	Axillary equation
	CF: $x=A\mathrm{e}^{-\frac{1}{2}t}+B\mathrm{e}^{\frac{1}{5}t}$	**A1**	
	PI: $x=p+qt\Rightarrow\dot{x}=q\Rightarrow\ddot{x}=0$	**M1**	Forms PI and differentiates.
	$3q-p-qt=t+2\Rightarrow q=-1, p=-5.$	**M1 A1**	Substitutes.
	GS: $x=A\mathrm{e}^{-\frac{1}{2}t}+B\mathrm{e}^{\frac{1}{5}t}-t-5$	**A1**	States general solution.
	$\dot{x}=-\frac{1}{2}A\mathrm{e}^{-\frac{1}{2}t}+\frac{1}{5}B\mathrm{e}^{\frac{1}{5}t}-1$	**M1**	Differentiates.
	$A+B=5$ $-\frac{1}{2}A+\frac{1}{5}B=1$	**M1**	Forms simultaneous equations.
	$A=0,\ B=5$	**A1**	
	$x=5\mathrm{e}^{\frac{1}{5}t}-t-5$	**A1**	States PS.
		10	

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Question	Answer	Marks	Guidance
8	$9u^2+6u+1=0 \Rightarrow (3u+1)^2=0$	**M1**	Auxiliary equation.
	CF: $x=(At+B)\mathrm{e}^{-\frac{1}{3}t}$	**A1**	States CF.
	$x=p\sin t+q\cos t \Rightarrow \dot{x}=p\cos t-q\sin t$ $\Rightarrow \ddot{x}=-p\sin t-q\cos t$	**M1**	Forms PI and differentiates.
	$-9p-6q+p=50$ $-9q+6p+q=0$	**M1**	Substitutes.
	$q=-3, p=-4.$	**A1**	
	$x=(At+B)\mathrm{e}^{-\frac{1}{3}t}-4\sin t-3\cos t$	**A1**	States general solution. FT if correct form of CF/PI.
	$\dot{x}=-\frac{1}{3}(At+B)\mathrm{e}^{-\frac{1}{3}t}+A\mathrm{e}^{-\frac{1}{3}t}-4\cos t+3\sin t$	**M1**	Differentiates and forms equations using initial conditions.
	$B=3$	**B1**	
	$-\frac{1}{3}B+A-4=0 \Rightarrow A=5$	**A1**	.
	$x=(5t+3)\mathrm{e}^{-\frac{1}{3}t}-4\sin t-3\cos t$	**A1**	States PS.
		10	

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Q18.

Question	Answer	Marks	Guidance
11E(i)	$w = \cos y \Rightarrow \dfrac{\mathrm{d}w}{\mathrm{d}x} = -\sin y \dfrac{\mathrm{d}y}{\mathrm{d}x}$	**B1**	
	$\dfrac{\mathrm{d}^2w}{\mathrm{d}x^2} = -\sin y \dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} - \cos y \left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)^2$	**B1**	
	$\dfrac{\mathrm{d}^2w}{\mathrm{d}x^2} + 2\dfrac{\mathrm{d}w}{\mathrm{d}x} + w = -\sin y \dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} - \cos y \left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)^2 - 2\sin y \dfrac{\mathrm{d}y}{\mathrm{d}x} + \cos y$	**M1**	Uses substitution to obtain w - x equation, AG.
	$= -\cos y\left(\mathrm{e}^{-2x}\sec y\right) = -\mathrm{e}^{-2x}$	**A1**	
		4	

Question	Answer	Marks	Guidance
11E(ii)	$m^2+2m+1=0 \Rightarrow m=-1$	**M1**	Finds CF.
	CF: $w=\left(Ax+B\right)\mathrm{e}^{-x}$	**A1**	
	PI: $w=k\mathrm{e}^{-2x} \Rightarrow w'=-2k\mathrm{e}^{-2x} \Rightarrow w''=4k\mathrm{e}^{-2x}$	**M1**	Forms PI and differentiates.
	$4k-4k+k=-1 \Rightarrow k=-1$	**A1**	
	$w=\left(Ax+B\right)\mathrm{e}^{-x}-\mathrm{e}^{-2x}$	**A1**	States general solution.
	$x=0$, $y=\frac{1}{3}\pi$, $w=\frac{1}{2} \Rightarrow B=\frac{3}{2}$	**B1**	Uses initial conditions to find constants.
	$w'=-\left(Ax+B\right)\mathrm{e}^{-x}+A\mathrm{e}^{-x}+2\mathrm{e}^{-2x}$	**M1**	Differentiates general solution.
	$x=0$, $y=\frac{1}{3}\pi$, $y'=\frac{\sqrt{3}}{3}$, $w'=-\frac{1}{2} \Rightarrow -\frac{1}{2}=-\frac{3}{2}+A+2 \Rightarrow A=-1$	**M1 A1**	Substitutes initial conditions.
	$y=\cos^{-1}\left(\left(\frac{3}{2}-x\right)\mathrm{e}^{-x}-\mathrm{e}^{-2x}\right)$	**A1**	States particular solution for y in terms of x.
		10	

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Question	Answer	Marks	Guidance
7	The auxiliary equation is $\left(n+\frac{1}{7}\right)^2=0\Rightarrow n=-\frac{1}{7}$.	**M1**	
	CF: $y=\left(A+Bx\right)e^{-\frac{1}{7}x}$.	**A1**	
	PI: $y=px+q$, $y'=p$ and $y''=0$.	**M1**	Correct form of PI and derivatives
	$14p+px+q=49x+735\Rightarrow p=49,\ q=49$.	**M1 A1**	Substitutes in equation correctly
	Thus $y=\left(A+Bx\right)e^{-\frac{1}{7}x}+49\left(x+1\right)$.	**A1 FT**	FT only on correct form of CF AEF
	$y=0$ when $x=0$ gives $A+49=0\Rightarrow A=-49$.	**B1**	
	$y'=-\frac{1}{7}\left(A+Bx\right)e^{-\frac{1}{7}x}+Be^{-\frac{1}{7}x}+49$.	**M1**	Differentiating their y
	$y'=0$ when $x=0$ gives $-\frac{1}{7}A+B+49=0\Rightarrow B=-56$	**A1**	AEF
	Thus $y=-\left(49+56x\right)e^{-\frac{1}{7}x}+49\left(x+1\right)$.	**A1**	
		10	

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Q20.

Question	Answer	Marks	Guidance
10(i)	$y' = x + x't$	**B1**	
	$y'' = x' + x' + tx''$	**B1**	
	Substitute correctly	**B1**	AG
10(ii)	Auxiliary equation: $m^2 + 9 = 0 \Rightarrow m = \pm 3i.$	**M1**	Correct auxiliary
	CF $= A\cos 3t + B\sin 3t.$	**A1**	

Question	Answer	Marks	Guidance
	PI $y = at^2 + bt + c$ so $y' = 2at + b$ and $y'' = 2a$	**M1**	Differentiate twice and substitute
	$a = \frac{1}{3}, b = 0, c = \frac{1}{27}.$	**A1**	
	$y = A\cos 3t + B\sin 3t + \frac{1}{3}t^2 + \frac{1}{27}$	**A1FT**	Their CF + their PI both in correct form
	$x = \frac{\pi}{9}$ when $t = \frac{\pi}{3}$ gives $A = \frac{1}{27}$.	**B1**	
	$y' = -3A\sin 3t + 3B\cos 3t + \frac{2}{3}t$	**M1**	Differentiating their y of equivalent difficulty
	$x' = \frac{2}{3}$ when $t = \frac{\pi}{3}$ gives $B = -\frac{\pi}{27}$.	**A1**	
	$x = \frac{\cos 3t - \pi\sin 3t + 9t^2 + 1}{27t}$	**A1**	AEF
		12	

Question	Answer	Marks	Guidance
10(i)	$m^2+2m+10=0 \Rightarrow m=-1\pm 3i.$	**M1**	Finds complementary function.
	$x=e^{-t}\left(A\cos 3t+B\sin 3t\right)$	**A1**	
	$x=p\cos 3t+q\sin 3t$	**M1**	Finds particular integral.
	$\Rightarrow \dot{x}=-3p\sin 3t+3q\cos 3t \Rightarrow \ddot{x}=-9p\cos 3t-9q\sin 3t$	**A1**	
	$\left(p+6q\right)\cos 3t+\left(q-6p\right)\sin 3t=37\sin 3t$	**M1**	
	$\Rightarrow \quad p=-6, \quad q=1$ $x=e^{-t}\left(A\cos 3t+B\sin 3t\right)-6\cos 3t+\sin 3t$	**A1**	
	$x=3$ when $t=0 \Rightarrow A-6=3 \Rightarrow A=9$	**B1**	Uses initial conditions.
	$\dot{x}=e^{-t}\left(-3A\sin 3t+3B\cos 3t\right)-e^{-t}\left(A\cos 3t+B\sin 3t\right)+18\sin 3t+3\cos 3t$	**M1 A1**	
	$\dot{x}=0$ when $t=0 \Rightarrow 3B-A+3=0 \Rightarrow B=2$ $x=e^{-t}\left(9\cos 3t+2\sin 3t\right)-6\cos 3t+\sin 3t$	**A1**	Obtains solution, AEF.
		10	
10(ii)	As $t\to\infty$, $e^{-t}\to 0 \Rightarrow x\approx -6\cos 3t+\sin 3t$	**B1**	Obtains limit.
	$\sin 3t-6\cos 3t=\sqrt{37}\left(\frac{1}{\sqrt{37}}\sin 3t-\frac{6}{\sqrt{37}}\cos 3t\right)$	**M1**	Converts to $R\sin\left(3t-\phi\right)$
	$=\sqrt{37}\sin\left(3t-\tan^{-1}6\right).$	**A1**	AG
		3	

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Question	Answer	Marks	Guidance
4(i)	$m^2+2m+1=0\Rightarrow(m+1)^2=0\Rightarrow m=-1$	**M1**	Forms and solves auxiliary equation.
	CF: $(A+Bt)e^{-t}$	**A1**	States CF.
	PI: $x=p\sin t+q\cos t$	**M1**	Uses correct form of PI and differentiates twice.
	$\Rightarrow\dot{x}=p\cos t-q\sin t\ \Rightarrow\ddot{x}=-p\sin t-q\cos t$	**A1**	
	$-p\sin t-q\cos t+2(p\cos t-q\sin t)+p\sin t+q\cos t=4sint$	**M1**	Compares coefficients and attempts to solve
	$2p=0\Rightarrow p=0$. $-2q=4\Rightarrow q=-2$.	**A1**	
	GS: $\boldsymbol{x}=(A+Bt)e^{-t}-2\cos t$	**A1FT**	States general solution. FT on correct form only
		7	
4(ii)	$x\approx-2\cos t$	**B1FT**	
		1	

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Q23.

10(i)	$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = 2t^{\frac{1}{2}} \frac{dy}{dt}$ (AG)	**B1**	
	$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right) \times \frac{dt}{dx} = \left\{ t^{-\frac{1}{2}} \frac{dy}{dt} + 2t^{\frac{1}{2}} \frac{d^2y}{dt^2} \right\} 2t^{\frac{1}{2}} = 2\frac{dy}{dt} + 4t\frac{d^2y}{dt^2}$ (AG)	**M1A1**	
	Total:	**3**	
10(ii)	Substitute in (*): $2\frac{dy}{dt} + 4t\frac{d^2y}{dt^2} - 16t\frac{dy}{dt} - 2\frac{dy}{dt} + 12ty = 4te^{-t}$ $\Rightarrow \frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 3y = e^{-t}$ (AG)	**B1**	
	Total:	**1**	

Question	Answer	Marks	Guidance
10(iii)	CF: $(m-1)(m-3) = 0$	**M1**	
	$y = Ae^t + Be^{3t}$	**A1**	
	PI: $y = ke^{-t} \Rightarrow y' = -ke^{-t} \Rightarrow y'' = ke^{-t}$	**M1**	
	$ke^{-t} + 4ke^{-t} + 3ke^{-t} = e^{-t}$	**M1**	
	$\Rightarrow k - \frac{1}{8}$	**A1**	
	GS: $y = Ae^t + Be^{3t} + \frac{1}{8}e^{-t}$	**A1 FT**	
	$y = Ae^{x^2} + Be^{3x^2} + \frac{1}{8}e^{-x^2}$	**A1 FT**	

Question	Answer	Marks	Guidance
8	$m^2 + 6m + 9 = 0 \Rightarrow m = -3$	**M1**	
	CF: $x = Ae^{-3t} + Bte^{-3t}$	**A1**	
	$x = pt^2 + qt + r \Rightarrow \dot{x} = 2pt + q$ and $\ddot{x} = 2p$	**M1**	
	$\Rightarrow 2p + 6(2pt + q) + 9(pt^2 + qt + r) = 18t^2 + 6t + 1$	**M1**	
	$\Rightarrow 9pt^2 + (9q + 12p)t + (2p + 6q + 9r) = 18t^2 + 6t + 1$ $\Rightarrow p = 2, q = -2, r = 1$ whence PI: $2t^2 - 2t + 1$	**A1**	
	GS: $x = Ae^{-3t} + Bte^{-3t} + 2t^2 - 2t + 1$	**A1FT**	
	$x = 3$ when $t = 0 \Rightarrow A = 2$	**B1**	
	$\dot{x} = -3Ae^{-3t} - 3Bte^{-3t} + Be^{-3t} + 4t - 2$	**M1**	
	$\dot{x} = 0$ when $t = 0 \Rightarrow B = 8$	**A1**	
	Hence $x = 2e^{-3t} + 8te^{-3t} + 2t^2 - 2t + 1$	**A1**	
	Total:	**10**	

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Q25.

Question	Answer	Marks	Guidance
2	CF: $m^2 + 2m + 5 = 0 \Rightarrow m = -1 \pm 2\text{i}$	**M1**	
	$e^{-t}(A \cos 2t + B \sin 2t)$	**A1**	
	PI: $x = pt^2 + qt + r \Rightarrow \dot{x} = 2pt + q \Rightarrow \ddot{x} = 2p$	**M1**	
	$2p + 4pt + 2q + 5pt^2 + 5qt + 5r = 4 - 5t^2$	**M1**	
	$\Rightarrow p = -1, q = \frac{4}{5}, r = \frac{22}{25}$	**A1**	
	GS: $x = e^{-t}(A \cos 2t + B \sin 2t) + \frac{22}{25} + \frac{4}{5}t - t^2$	**A1FT**	
		6	

Q26.

9	$y' = 2kx\mathrm{e}^{2x} + 2kx^2\mathrm{e}^{2x}$ $y'' = 2k\mathrm{e}^{2x} + 8kx\mathrm{e}^{2x} + 4kx^2\mathrm{e}^{2x}$ $2k\mathrm{e}^{2x} + 8kx\mathrm{e}^{2x} + 4kx^2\mathrm{e}^{2x} - 8kx\mathrm{e}^{2x} - 8kx^2\mathrm{e}^{2x} + 4kx^2\mathrm{e}^{2x} = 4\mathrm{e}^{2x}$ $\Rightarrow 2k = 4 \Rightarrow k = 2$ $m^2 - 4m + 4 = 0 \Rightarrow (m-2)^2 = 0 \Rightarrow m = 2$ CF: $y = A\mathrm{e}^{2x} + Bx\mathrm{e}^{2x}$ $y = A\mathrm{e}^{2x} + Bx\mathrm{e}^{2x} + 2x^2\mathrm{e}^{2x}$ $y = 3$ when $x = 0 \Rightarrow A = 3$ $y' = 2A\mathrm{e}^{2x} + B\left(\mathrm{e}^{2x} + 2x\mathrm{e}^{2x}\right) + \left(4x\mathrm{e}^{2x} + 4x^2\mathrm{e}^{2x}\right)$ $y' = -2$ when $x = 0$ and A $= 3 \Rightarrow B = -8$ $\Rightarrow y = 3\mathrm{e}^{2x} - 8x\mathrm{e}^{2x} + 2x^2\mathrm{e}^{2x}$	**B1** **B1**√ **M1** **A1** [4] **M1** **A1** **A1**√ [3] **B1** **M1** **A1** **A1**√ [4]

Q27.

Qu	Solution	Marks
10	$x\dfrac{\mathrm{d}y}{\mathrm{d}x} = x\dfrac{\mathrm{d}y}{\mathrm{d}u}\times\dfrac{\mathrm{d}u}{\mathrm{d}x} = x\dfrac{\mathrm{d}y}{\mathrm{d}u}\times\dfrac{1}{x} = \dfrac{\mathrm{d}y}{\mathrm{d}u}$ **(AG)**	**B1**
	$x^2\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = x^2\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right) = x^2\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{1}{x}\dfrac{\mathrm{d}y}{\mathrm{d}u}\right)$	**M1**
	$= x^2\left(-\dfrac{1}{x^2}\dfrac{\mathrm{d}y}{\mathrm{d}u} + \dfrac{1}{x}\dfrac{\mathrm{d}}{\mathrm{d}u}\left[\dfrac{\mathrm{d}y}{\mathrm{d}u}\right]\dfrac{\mathrm{d}u}{\mathrm{d}x}\right)$	
	$= -\dfrac{\mathrm{d}y}{\mathrm{d}u} + x\dfrac{\mathrm{d}^2y}{\mathrm{d}u^2}.\dfrac{1}{x} = \dfrac{\mathrm{d}^2y}{\mathrm{d}u^2} - \dfrac{\mathrm{d}y}{\mathrm{d}u}$ **(AG)**	**A1** [3]
	Substituting in differential equation:	
	$\dfrac{\mathrm{d}^2y}{\mathrm{d}u^2} - \dfrac{\mathrm{d}y}{\mathrm{d}u} + 3\dfrac{\mathrm{d}y}{\mathrm{d}u} + 17y = 34u + 21 \Rightarrow \dfrac{\mathrm{d}^2y}{\mathrm{d}u^2} + 2\dfrac{\mathrm{d}y}{\mathrm{d}u} + 17y = 34u + 21$ **(AG)**	**B1** [1]
	$m^2 + 2m + 17 = 0 \Rightarrow m = -1 \pm 4\mathrm{i} \Rightarrow y_c = \mathrm{e}^{-u}\left(A\cos 4u + B\sin 4u\right)$	**M1A1**
	$y_p = cu + d \Rightarrow y' = c$ and $y'' = 0 \Rightarrow 2c + 17cu + 17d \equiv 34u + 21$	**M1**
	$\Rightarrow c = 2 \quad , \quad d = 1 \Rightarrow y_p = 2u + 1$	**A1**
	$\Rightarrow y = \dfrac{1}{x}\left(A\cos\left[4\ln x\right] + B\sin\left[4\ln x\right]\right) + 2\ln x + 1$ (or in terms of u)	**A1**√
	$y = 0$ when $x = 1 \Rightarrow 0 = A + 1 \Rightarrow A = -1$ **or** u = 0, y= 0 etc	**B1**
	$\dfrac{\mathrm{d}y}{\mathrm{d}x} = -\dfrac{1}{x^2}\left(A\cos\left[4\ln x\right] + B\sin\left[4\ln x\right]\right) + \dfrac{1}{x}\left(-A\sin\left[4\ln x\right].\dfrac{4}{x} + B\cos\left[4\ln x\right].\dfrac{4}{x}\right) + \dfrac{2}{x}$	**M1**
	$\dfrac{\mathrm{d}y}{\mathrm{d}x} = -1$ when $x = 1 \Rightarrow -1 = 1 + 4B + 2 \Rightarrow B = -1$	**A1**
	or all in terms of u	
	Hence $y = 2\ln x + 1 - \dfrac{1}{x}\left(\cos\left[4\ln x\right] + \sin\left[4\ln x\right]\right)$ (oe)	**A1** [9]

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Q28.

6	$m^2+7m+10=0 \Rightarrow (m+2)(m+5)=0 \Rightarrow m=-2 \text{ or } -5$	**M1**	
	CF: $Ae^{-2t}+Be^{-5t}$	**A1**	
	PI: $x=p\sin 2t+q\cos 2t \Rightarrow \dot{x}=2p\cos 2t-2q\sin 2t \Rightarrow \ddot{x}=-4p\sin 2t-4q\cos 2t$	**M1A1**	
	Substituting: $14p+6q=0$ $6p-14q=116$	**M1**	
	Solving: $p=3$, $q=-7 \Rightarrow x=3\sin 2t-7\cos 2t$	**M1A1**	
	GS: $x=Ae^{-2t}+Be^{-5t}+3\sin 2t-7\cos 2t$	**A1**√	[8]
	For large positive values of t $x \approx 3\sin 2t-7\cos 2t$.	**B1**√	[1]

Q29.

11 (e)	$\frac{dv}{dx}=-\frac{1}{y^2}\frac{dy}{dx}$	B1
	$\frac{d^2v}{dx^2}=-\frac{1}{y^2}\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2\frac{d^2v}{dy^2}=-\frac{1}{y^2}\frac{d^2y}{dx^2}+\frac{2}{y^3}\left(\frac{dy}{dx}\right)^2$	M1A1
	$\frac{2}{y^3}\left(\frac{dy}{dx}\right)^2-\frac{1}{y^2}\frac{d^2y}{dx^2}-\frac{2}{y^2}\frac{dy}{dx}+\frac{5}{y}=17+6x-5x^2 \sim \frac{d^2v}{dx^2}+2\frac{dv}{dx}+5v=17+6x-5x^2$ (AG)	A1 (4)
	$m^2+2m+5=0 \Rightarrow m=-1\pm 2i \Rightarrow CF: e^{-x}(A\cos 2x+B\sin 2x)$	M1A1
	PI : $v=px^2+qx+r \Rightarrow v'=2px+q \Rightarrow v''=2p$	M1
	Equate coefficients: $5p=-5$ $4p+5q=6$ $2p+2q+5r=17$	M1
	$p=-1$, $q=2$, $r=3$	A1
	GS : $v=e^{-x}(A\cos 2x+B\sin 2x)+3+2x-x^2$	A1
	When $x=0$ $y=\frac{1}{2} \Rightarrow v=2$ and $\frac{dy}{dx}=-1 \Rightarrow \frac{dv}{dx}=4$	

	$\Rightarrow 2 = A + 3 \Rightarrow A = -1$	B1
	$v' = e^{-x}\ (-2A \sin 2x + 2B \cos 2x) - e^{-x}\ (A \cos 2x + B \sin 2x)$	M1
	$\Rightarrow 4 = 2B + 1 + 2 \Rightarrow 2B = 1 \Rightarrow B = \frac{1}{2}$	A1
	$\Rightarrow y = \left[e^{-x}\left(\frac{1}{2}\sin 2x - \cos 2x\right) + 3 + 2x - x^2 \right]^{-1}$	A1 (10) **Total: 14**

Q30.

9	$(m-5)(m+2) = 0 \Rightarrow \text{CF} : x = A\text{e}^{5t} + B\text{e}^{-2t}$	M1A1
	$\text{PI} : x = p \sin t + q \cos t \Rightarrow \dot{x} = p \cos t - q \sin t \Rightarrow \ddot{x} = -p \sin t - q \cos t$	M1
	$\Rightarrow -p \sin t - q \cos t - 3p \cos t + 3q \sin t - 10p \sin t - 10q \cos t = 2 \sin t - 3 \cos t$	A1
	$\Rightarrow -11p + 3q = 2$ and $3p + 11q = 3 \Rightarrow p = -0.1,\ q = 0.3$	dM1A1
	$\text{GS: } x = A\text{e}^{5t} + B\text{e}^{-2t} + 0.3 \cos t - 0.1 \sin t$ CAO	A1
	Initial conditions $\Rightarrow A + B + 0.3 = 3.3 \Rightarrow A + B = 3$	B1
	and from $\dot{x} = 5A\text{e}^{5t} - 2B\text{e}^{-2t} - 0.3 \sin t - 0.1 \cos t$	M1
	$5A - 2B - 0.1 = 0.9 \Rightarrow 5A - 2B = 1$	A1
	$A = 1,\ B = 2 \Rightarrow x = \text{e}^{5t} + 2\text{e}^{-2t} + 0.3 \cos t - 0.1 \sin t$ CAO	A1 (11) **Total 11**

Q31.

2	$m^2+4m+4=0 \Rightarrow (m+2)^2=0 \Rightarrow m=-2$	**M1**
	CF: $Ae^{-2t}+Bte^{-2t}$ soi	**A1**
	PI : $x=pt^2+qt+r \Rightarrow \dot{x}=2pt+q \Rightarrow \ddot{x}=2p$	**M1**
	$\Rightarrow 2p+8pt+4q+4pt^2+4qt+4r=7-2t^2$	**M1**
	$\Rightarrow p=-\frac{1}{2},\ q=1,\ r=1$	**A1**
	GS: $x=Ae^{-2t}+Bte^{-2t}-\frac{1}{2}t^2+t+1$	**A1** [6] **Total 6**

Q32.

4	$m^2-\ m+25=0 \Rightarrow m=3\pm 4\text{i}$	M1
	CF: $x=e^{3t}(A\cos 4t+B\sin 4t)$	A1
	PI: $x=p\sin 2t+q\cos 2t \Rightarrow \dot{x}=2p\cos 2t-2q\sin 2t \Rightarrow \ddot{x}=-4p\sin 2t-4q\cos 2t$	M1
	$\Rightarrow 21p+12q=195$	
	$-12p+21q=0$	M1
	$\Rightarrow p=7, q=4$	A1
	GS: $x=e^{3t}(A\cos 4t+B\sin 4t)+7\sin 2t+4\cos 2t$	A1 [6]

Q33.

10	$m^2+0.\quad m+0.0064=0 \Rightarrow (m+0.08)^2=0 \Rightarrow m=-0.08$	M1
	CF: $x=(A+Bt)e^{-0.08t}$	A1
	PI: $x=p+qt \Rightarrow \dot{x}=q \Rightarrow \ddot{x}=0$	M1
	$0.16q+0.0064(p+qt)=8.64+0.32t \Rightarrow p=100\ ,\ q=50$.	M1A1
	$x=(A+Bt)e^{-0.08t}+100+50t$	A1
	$x=0$ when $t=0 \Rightarrow A=-100$	B1
	$\dot{x}=-0.08(Bt-100)e^{-0.08t}+Be^{-0.08t}+50$ (*) (Correct form req. for M mark.)	M1
	$\dot{x}=0$ when $t=0 \Rightarrow B=-58$	A1
	$x=100+50t-(100+58t)e^{-0.08t}$	A1 [10]
	From (*) $e^{-0.08t}\to 0$ as $t\to\infty$ (Correct form req. for M mark.)	M1
	$\therefore \dot{x}\to 50$	A1 [2]

Q34.

9	$v = xy \Rightarrow \dfrac{\mathrm{d}v}{\mathrm{d}x} = y + x\dfrac{\mathrm{d}y}{\mathrm{d}x} \Rightarrow \dfrac{\mathrm{d}^2v}{\mathrm{d}x^2} = 2\dfrac{\mathrm{d}y}{\mathrm{d}x} + x\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$	B1B1
	$x\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} + (2x+2)\dfrac{\mathrm{d}y}{\mathrm{d}x} + (2-3x)y = 10\mathrm{e}^{2x}$ $\Rightarrow 2\dfrac{\mathrm{d}y}{\mathrm{d}x} + x\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} + 2\left(y + x\dfrac{\mathrm{d}y}{\mathrm{d}x}\right) - 3xy = 10e^{2x} \Rightarrow \dfrac{\mathrm{d}^2v}{\mathrm{d}x^2} + 2\dfrac{\mathrm{d}v}{\mathrm{d}x} - 3v = 10\mathrm{e}^{2x}$	M1A1 (4)
	$m^2 + 2m - 3 = 0 \Rightarrow (m+3)(m-1) = 0 \Rightarrow m = -3, 1$	M1
	CF: $A\mathrm{e}^{-3x} + B\mathrm{e}^{x}$	A1
	$v = k\mathrm{e}^{2x} \Rightarrow v' = 2k\mathrm{e}^{2x} \Rightarrow v'' = 4k\mathrm{e}^{2x}$	M1
	$\Rightarrow 4k\mathrm{e}^{2x} + 4k\mathrm{e}^{2x} - 3k\mathrm{e}^{2x} = 10\mathrm{e}^{2x} \Rightarrow 5k = 10 \Rightarrow k = 2 \Rightarrow \mathrm{PI}: 2\mathrm{e}^{2x}$	M1A1
	GS: $v = A\mathrm{e}^{-3x} + B\mathrm{e}^{x} + 2\mathrm{e}^{2x} \Rightarrow y = \dfrac{1}{x}\left(A\mathrm{e}^{-3x} + B\mathrm{e}^{x} + 2\mathrm{e}^{2x}\right)$	M1A1 (7) **[11]**